

ε -Kernel Coresets for Stochastic Points

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Abstract

With the dramatic growth in the number of application domains that generate probabilistic, noisy and uncertain data, there has been an increasing interest in designing algorithms for geometric or combinatorial optimization problems over such data. In this paper, we initiate the study of constructing ε -kernel coresets for uncertain points. We consider uncertainty in the existential model where each point's location is fixed but only occurs with a certain probability, and the locational model where each point has a probability distribution describing its location. An ε -kernel coreset approximates the width of a point set in any direction. We consider approximating the expected width (an ε -EXP-KERNEL), as well as the probability distribution on the width (an (ε, τ) -QUANT-KERNEL) for any direction. We show that there exists a set of $O(\varepsilon^{-(d-1)/2})$ deterministic points which approximate the expected width under the existential and locational models, and we provide efficient algorithms for constructing such coresets. We show, however, it is not always possible to find a subset of the original uncertain points which provides such an approximation. However, if the existential probability of each point is lower bounded by a constant, an ε -EXP-KERNEL is still possible. We also provide efficient algorithms for construct an (ε, τ) -QUANT-KERNEL coreset in nearly linear time. Our techniques utilize or connect to several important notions in probability and geometry, such as Kolmogorov distances, VC uniform convergence and Tukey depth, and may be useful in other geometric optimization problem in stochastic settings. Finally, combining with known techniques, we show a few applications to approximating the extent of uncertain functions, maintaining extent measures for stochastic moving points and some shape fitting problems under uncertainty.

1 Introduction

Uncertain Data Models: The wide deployment of sensor monitoring infrastructure and increasing prevalence of technologies such as data integration and cleaning [19, 25] have

resulted in an abundance of uncertain, noisy and probabilistic data in many scientific and application domains. Managing, analyzing, and solving optimization problems over such data have become an increasingly important issue and have attracted significant attentions from several research communities including theoretical computer science, databases, machine learning and wireless networks.

In this paper, we focus on two stochastic models, the existential model and locational model. Both models have been studied extensively for a variety of computational geometry problems or combinatorial problems, such as closest pairs [45], nearest neighbors [5, 45], minimum spanning trees [42, 46], convex hulls [6, 33, 61, 64], maxima [3], perfect matchings [42], clustering [20, 35], minimum enclosing balls [55] and range queries [1, 4, 49].

1. Existential uncertainty model: In this model, there are a set \mathcal{P} of n points in \mathbb{R}^d . Throughout this paper, we assume that the dimension d is a constant. Each point $v \in \mathcal{P}$ is associated with a real number (called *existential probability*) $p_v \in [0, 1]$ which indicates that v is present independently with probability p_v .
2. Locational uncertainty model: There are a set \mathcal{P} of n points and the existence of each point is certain. However, the location of each point $v \in \mathcal{P}$ is a random location in \mathbb{R}^d . We assume the probability distribution is discrete and independent of other points. For a point $v \in \mathcal{P}$ and a location $s \in \mathbb{R}^d$, we use $p_{v,s}$ to denote the probability that the location of point v is s . Let S be the set of all possible locations, and let $|S| = m$ be the number of all such locations.

In the locational uncertainty model, we distinguish the use of the terms “points” and “locations”; a point refers to the object with uncertain locations and a location refers to a point (in the usual sense) in \mathbb{R}^d . We will use capital letters (e.g., P, S, \dots) to denote sets of deterministic points and calligraphy letters ($\mathcal{P}, \mathcal{S}, \dots$) to denote sets of stochastic points.

Coresets: Given a large dataset P and a class C of queries, a coreset S is a dataset of much smaller size such that for every query $r \in C$, the answer $r(S)$ for the small dataset S is close to the answer $r(P)$ for the original large dataset P . Coresets [58] have become more relevant in the era of big data as they can drastically reduce the size of a dataset while guaranteeing that answers for certain queries are provably close. An early notion of a coreset concerns the directional width problem (in which a coreset is called an ε -kernel) and several other geometric shape-fitting problems in the seminal paper [7].

We introduce some notations and review the definition of ε -kernel. We assume the dimension d is a constant. For a set P of deterministic points, the *support function* $f(P, u)$ is defined to be $f(P, u) = \max_{p \in P} \langle u, p \rangle$ for $u \in \mathbb{R}^d$, where $\langle \cdot, \cdot \rangle$ is the inner product. The *directional width* of P in direction $u \in \mathbb{R}^d$, denoted by $\omega(P, u)$, is defined by $\omega(P, u) = f(P, u) + f(P, -u)$. It is not hard to see that the support function and the directional width only depend on the convex hull of P . A subset $Q \subseteq P$ is called an ε -kernel of P if for each direction $u \in \mathbb{R}^d$, $(1 - \varepsilon)\omega(P, u) \leq \omega(Q, u) \leq \omega(P, u)$. For any set of n points, there is an ε -kernel of size $O(\varepsilon^{-(d-1)/2})$ [7, 8], which can be constructed in $O(n + \varepsilon^{-(d-3/2)})$ time [17, 65].

1.1 Problem Formulations

In this paper we focus on constructing ε -kernel coresets when the input data is uncertain. These results not only provide an understanding of how to compactly represent an approximate convex hull under uncertainty, but can lead to solutions to a variety of other shape-fitting problems.

ε -Kernels for expectations. Suppose \mathcal{P} is a set of stochastic points (in either the existential or locational uncertainty model). Define the *expected* directional width of \mathcal{P} in direction u to be $\omega(\mathcal{P}, u) = \mathbb{E}_{P \sim \mathcal{P}}[\omega(P, u)]$, where $P \sim \mathcal{P}$ means that P is a (random) realization of \mathcal{P} .

Definition 1. For a constant $\varepsilon > 0$, a set S of (deterministic or stochastic) points in \mathbb{R}^d is called an ε -EXP-KERNEL of \mathcal{P} , if for all directions $u \in \mathbb{R}^d$,

$$(1 - \varepsilon)\omega(\mathcal{P}, u) \leq \omega(S, u) \leq \omega(\mathcal{P}, u).$$

Recall in the deterministic setting, we require that the ε -kernel S be a subset of the original point set (we call this *the subset constraint*). It is important to consider the subset constraint since it can reveal how concisely arbitrary uncertain point sets can be represented with just a few uncertain points (size depending only on ε). For ε -kernels on deterministic points, $\Omega(\varepsilon^{-(d-1)/2})$ points may be required and can always be found under the subset constraint [7, 8]. However, in the stochastic setting, we will show this is no longer true. Coresets without the subset constraint, in fact made of deterministic points, can sometimes be obtained when no coreset with the subset constraint is possible.

ε -Kernels for probability distributions. Sometimes it is useful to obtain more than just the expected value (say of the width) on a query; rather one may want (an approximation of) a representation of the full probability distribution that the query can take.

Definition 2. For a constant $\varepsilon, \tau > 0$, a set \mathcal{S} of stochastic points in \mathbb{R}^d is called an (ε, τ) -QUANT-KERNEL of \mathcal{P} , if for all directions u and all $x \geq 0$,

$$\Pr_{P \sim \mathcal{P}}[\omega(P, u) \leq (1 - \varepsilon)x] - \tau \leq \Pr_{S \sim \mathcal{S}}[\omega(S, u) \leq x] \leq \Pr_{P \sim \mathcal{P}}[\omega(P, u) \leq (1 + \varepsilon)x] + \tau. \quad (1)$$

In the above definition, we do not require the points in \mathcal{S} are independent. So when they are correlated, we will specify the distribution of \mathcal{S} . If all points in \mathcal{P} are deterministic and $\tau < 0.5$, the above definition essentially boils down to requiring $(1 - \varepsilon)\omega(\mathcal{P}, u) \leq \omega(\mathcal{S}, u) \leq (1 + \varepsilon)\omega(\mathcal{P}, u)$. Assuming the coordinates of the input points are bounded, an (ε, τ) -QUANT-KERNEL ensures that for any choice of u , the cumulative distribution function of $\omega(\mathcal{S}, u)$ is within a distance ε under the Lévy metric, to that of $\omega(\mathcal{P}, u)$.¹

ε -Kernels for expected fractional powers. Sometimes, the notion ε -EXP-KERNEL is not powerful enough for certain shape fitting problems (e.g., the minimum enclosing cylinder

¹ Assuming the coordinates of the input points are bounded, the requirement for an (ε, τ) -QUANT-KERNEL is in fact stronger than that of Lévy distance being no larger than ε as the former requires a multiplicative error on length, which gives better guarantee when the length is small.

problem and the minimum spherical shell problem) in the stochastic setting. The main reason is the appearance of the l_2 -norm in the objective function. So we need to be able to handle the fractional powers in the objective function. For a set P of points in \mathbb{R}^d , the polar set of P is defined to be $P^\star = \{u \in \mathbb{R}^d \mid \langle u, v \rangle \geq 0, \forall v \in P\}$. Let r be a positive integer. Given a set P of points in \mathbb{R}^d and $u \in P^\star$, we define a function

$$T_r(P, u) = \max_{v \in P} \langle u, v \rangle^{1/r} - \min_{v \in P} \langle u, v \rangle^{1/r}.$$

We only care about the directions in \mathcal{P}^\star (i.e., the polar of the points in \mathcal{P}) for which $T_r(P, u), \forall P \sim \mathcal{P}$ is well defined.

Definition 3. For a constant $\varepsilon > 0$, a positive integer r , a set \mathcal{S} of stochastic points in \mathbb{R}^d is called an (ε, r) -FPOW-KERNEL of \mathcal{P} , if for all directions $u \in \mathcal{P}^\star$,

$$(1 - \varepsilon)\mathbb{E}_{P \sim \mathcal{P}}[T_r(P, u)] \leq \mathbb{E}_{P \sim \mathcal{S}}[T_r(P, u)] \leq (1 + \varepsilon)\mathbb{E}_{P \sim \mathcal{P}}[T_r(P, u)].$$

1.2 Our Results

Now, we discuss the main technical results of the paper.

ε -Kernels for expectations. First, we consider ε -EXP-KERNELS under various constraints. Our first main result is that an ε -EXP-KERNEL of size $O(\varepsilon^{-(d-1)/2})$ exists for both existential and locational uncertainty model and can be constructed in nearly linear time.

Theorem 4. \mathcal{P} is a set of n uncertain points in \mathbb{R}^d (in either locational uncertainty model or existential uncertainty model). There exists an ε -EXP-KERNEL of size $O(\varepsilon^{-(d-1)/2})$ for \mathcal{P} . For existential uncertainty model (locational uncertainty model resp.), such an ε -EXP-KERNEL can be constructed in $O(\varepsilon^{-(d-1)}n \log n)$ time, ($O(\varepsilon^{-(d-1)}m \log m)$ time resp.), where n is the number of points and m is the total number of possible locations.

The existential result is a simple Minkowski sum argument. We first show that there exists a convex polytope M such that for any direction, the directional width of M is exactly the same as the expected directional width of \mathcal{P} (Lemma 10). This immediately implies the existence of a ε -EXP-KERNEL consisting $O(\varepsilon^{-(d-1)/2})$ deterministic points (using the result in [7]), but without the subset constraint. The Minkowski sum argument seems to suggest that the complexity of M is exponential. However, we show that the complexity of M is in fact polynomial $O(n^{2d-2})$ and we can construct it explicitly in $O(n^{2d-1} \log n)$ time (Theorem 14).

Although the complexity of M is polynomial, we cannot afford to construct it explicitly if we are to construct an ε -EXP-KERNEL in nearly linear time. Thus we construct the ε -EXP-KERNEL without explicitly constructing M . In particular, we show that it is possible to find the extreme vertex of M in a given direction in nearly linear time, by computing the gradient of the support function of M . We also provide quadratic-size data structures that can calculate the exact width $\omega(\mathcal{P}, \cdot)$ in logarithmic time under both models in \mathbb{R}^2 (Appendix D).

We also show that under subset constraint (i.e., the ε -EXP-KERNEL is required to be a subset of the original point set, with the same probability distribution for each chosen point), there is no ε -EXP-KERNEL of sublinear size (Lemma 19). However, if there is a constant lower bound $\beta > 0$ on the existential probabilities (called β -assumption), we can construct an ε -EXP-KERNEL of constant size (Theorem 20 and Appendix A).

ε -Kernels for probability distributions. Now, we describe our main results for (ε, τ) -QUANT-KERNELS. We first propose a quite simple but general algorithm for constructing (ε, τ) -QUANT-KERNELS, which achieves the following guarantee.

Theorem 5. *An (ε, τ) -QUANT-KERNEL of size $\tilde{O}(\tau^{-2}\varepsilon^{-3(d-1)/2})$ can be constructed in $\tilde{O}(n\tau^{-2}\varepsilon^{-(d-1)})$ time, under both existential and locational uncertainty models.*

The algorithm is surprisingly simple. Take a certain number of i.i.d. realizations, compute an ε -kernel for each realization, and then associate each kernel with probability $1/N$ (so the points are not independent). The analysis requires the VC uniform convergence bound for unions of halfspaces. The details can be found in Section 3.1.

For existential uncertainty model, we can improve the size bound as follows.

Theorem 6. *\mathcal{P} is a set of uncertain points in \mathbb{R}^d with existential uncertainty. Let $\lambda = \sum_{v \in \mathcal{P}} (-\ln(1 - p_v))$. There exists an (ε, τ) -QUANT-KERNEL for \mathcal{P} , which consists of a set of independent uncertain points of cardinality $\min\{\tilde{O}(\tau^{-2} \max\{\lambda^2, \lambda^4\}), \tilde{O}(\varepsilon^{-(d-1)}\tau^{-2})\}$. The algorithm for constructing such a coresset runs in $\tilde{O}(n \log^{O(d)} n)$ time.*

We note that another advantage of the improved construction is that the (ε, τ) -QUANT-KERNEL is a set of independent stochastic points (rather than correlated points as in Theorem 5). We achieve the improvement by two algorithms. The first algorithm transforms the Bernoulli distributed variables into Poisson distributed random variables and creates a probability distribution using the parameters of the Poissons, from which we take a number of i.i.d. samples as the coresset. Our analysis leverages the additivity of Poisson distributions and the VC uniform convergence bound (for halfspaces). However, the number of samples required depends on $\lambda(\mathcal{P})$, so the first algorithm only works when $\lambda(\mathcal{P})$ is small. The second algorithm complements the first one by identifying a convex set K that lies in the convex hull of \mathcal{P} with high probability (K exists when $\lambda(\mathcal{P})$ is large) and uses a small size deterministic ε -kernel to approximate K . The points in $\bar{K} = \mathcal{P} \setminus K$ can be approximated using the same sampling algorithm as in the first algorithm and we can show that $\lambda(\bar{K})$ is small, thus requiring only a small number of samples. Our algorithm can be easily extended to \mathbb{R}^d for any constant d and the size of the coresset is $\tilde{O}(\tau^{-2}\varepsilon^{-(d-1)})$. In Section 3.2.3, we show such an (ε, τ) -QUANT-KERNEL can be computed in $O(n \text{polylog} n)$ time using an iterative sampling algorithm. Our technique has some interesting connections to other important geometric problems (such as the Tukey depth problem) [54], may be interesting in its own right.

ε -Kernels for expected fractional powers. For (ε, r) -FPOW-KERNELS, we provide a linear time algorithm for constructing an (ε, r) -FPOW-KERNEL of size $\tilde{O}(\varepsilon^{-(rd-r+2)})$ in the existential uncertainty model under the β -assumption. The algorithm is almost the same as the construction in Section 3.1 except that some parameters are different.

Theorem 7. (Section 4) An (ε, r) -FPOW-KERNEL of size $\tilde{O}(\varepsilon^{-(rd-r+2)})$ can be constructed in $\tilde{O}(n\varepsilon^{-(rd-r+4)/2})$ time in the existential uncertainty model under the β -assumption.

Applications to Uncertain Function Approximation and Shape Fitting. Finally, we show that the above results, combined with the duality and linearization arguments [7], can be used to obtain constant size coresets for the function extent problem in the stochastic setting, and to maintain extent measures for stochastic moving points.

Using the above results, we also obtain efficient approximation schemes for various shape-fitting problems in the stochastic setting, such as minimum enclosing ball, minimum spherical shell, minimum enclosing cylinder and minimum cylindrical shell in different stochastic settings. We summarize our application results in the following theorems. The details can be found in Section 5.

Theorem 8. Suppose \mathcal{P} is a set of n independent stochastic points in \mathbb{R}^d under either existential or locational uncertainty model. There are linear time approximation schemes for the following problems: (1) finding a center point c to minimize $\mathbb{E}[\max_{v \in \mathcal{P}} \|v - c\|^2]$; (2) finding a center point c to minimize $\mathbb{E}[\text{obj}(c)] = \mathbb{E}[\max_{v \in \mathcal{P}} \|v - c\|^2 - \min_{v \in \mathcal{P}} \|v - c\|^2]$. Note that when $d = 2$ the above two problems correspond to minimizing the expected areas of the enclosing ball and the enclosing annulus, respectively.

Under β -assumption, we can obtain efficient approximation schemes for the following shape fitting problems.

Theorem 9. Suppose \mathcal{P} is a set of n independent stochastic points in \mathbb{R}^d , each appearing with probability at least β , for some fixed constant $\beta > 0$. There are linear time approximation schemes for minimizing the expected radius (or width) for the minimum spherical shell, minimum enclosing cylinder, minimum cylindrical shell problems over \mathcal{P} .

1.3 Other Related Work

Besides the stochastic models mentioned above, geometric uncertain data has also been studied in the *imprecise* model [13, 41, 47, 52, 56, 57, 62]. In this model, each point is provided with a region where it might be. This originated with the study of imprecision in data representation [36, 59], and can be used to provide upper and lower bounds on several geometric constructs such as the diameter, convex hull, and flow on terrains [26, 62].

Convex hulls have been studied for uncertain points: upper and lower bounds are provided under the imprecise model [29, 53, 56, 62], distributions of circumference and volume are calculated in the locational model [44, 51], the most likely convex hull is found in the existential model in \mathbb{R}^2 and shown NP-hard for \mathbb{R}^d for $d > 2$ and in the locational model [61], and the probability a query point is inside the convex hull [6, 33, 64]. As far as we know, the expected complexity of the convex hull under uncertain points has not been studied, although it has been studied [37] under other random data models.

There is a large body of literature [58] on constructing coresets for various problems, such as shape fitting [7, 8], shape fitting with outliers [40], clustering [18, 31, 32, 38, 48],

integrals [48], matrix approximation and regression [24, 31] and in different settings, such as geometric data streaming [8, 17] and privacy setting [30]. Coresets were constructed for imprecise points [53] to help derive results for approximating convex hulls and a variety of other shape-fitting problems, but because of the difference in models, these approaches do not translate to existential or locational models. In the locational model, coresets are created for range counting queries [1] under the subset constraint, but again these techniques do not translate because ε -kernel coresets in general cannot be constructed from a density-preserving subset of the data, as is preserved for the range counting coresets. Also in the locational model (and directly translating to the existential model) Löffler and Phillips [51] show how a large set of uncertain points can be approximated with a set of deterministic point sets, where each certain point set can be an ε -kernel. This can provide approximations similar to the (ε, τ) -QUANT-KERNEL with space $O(\varepsilon^{-(d+3)/2} \log(1/\delta))$ with probability at least $1 - \delta$. However it is not a coreset of the data, and answering width queries requires querying $O(\varepsilon^{-2} \log(1/\delta))$ deterministic point sets.

Recently, Munteanu et al. [55] studied the minimum enclosing ball problem over stochastic points, and obtained an efficient approximation scheme. Their algorithm and analysis utilize the results from the deterministic coreset literature [2]. However, they do not directly address the problem of constructing coresets for stochastic points and it is also unclear how to extend their technique to other shape fitting problems, such as minimum spherical shells.

Technically, our (ε, τ) -QUANT-KERNEL construction bears some similarity to the coreset by Har-Peled and Wang [40] for handling outliers. From the dual (function extent) perspective, they want to approximate the distance between two level sets in an arrangement of hyperplanes, and (the dual of) \mathcal{H} in Section 3.2.3 also needs to be (approximately) sandwiched by two fractional level sets (our hyperplanes have weights). However, we have an important requirement that the total weight outside $(1 + \varepsilon)\mathcal{H}$ must be small, which cannot be addressed by their technique.

2 ε -Kernels for Expectations of Width

We first state our results in this section for the existential uncertainty model. All results can be extended to the locational uncertainty model, with slightly different bounds (essentially replacing the number of points n with the number of locations m) or assumptions. We describe the difference for locational model in the appendix.

For simplicity of exposition, we assume in this section that all points in \mathcal{P} are in general positions and all p_v s are strictly between 0 and 1. For any $u, v \in \mathbb{R}^d$, we use $\langle u, v \rangle$ to denote the usual inner product $\sum_{i=1}^d u_i v_i$. For ease of notation, we write $v \succ_u w$ as a shorthand notation for $\langle u, v \rangle > \langle u, w \rangle$. For any $u \in \mathbb{R}^d$, the binary relation \succ_u defines a total order of all vertices in \mathcal{P} . (Ties should be broken in an arbitrary but consistent manner.) We call this order the *canonical order of \mathcal{P} with respect to u* . For any two points u and v , we use $d(u, v)$ or $\|v - u\|$ to denote their Euclidean distance. For any two sets of points, A and B , the Minkowski sum of A and B is defined as $A \oplus B := \{a + b \mid a \in A, b \in B\}$. Recall the definitions for a set P of deterministic points and a direction $u \in \mathbb{R}^d$, the support function

is $f(P, u) = \max_{p \in P} \langle u, p \rangle$ and the *directional width* is $\omega(P, u) = f(P, u) - f(P, -u)$. The support function and the directional width only depend on the convex hull of P .

Lemma 10. *Consider a set \mathcal{P} of uncertain points in \mathbb{R}^d (in either locational uncertainty model or existential uncertainty model). There exists a set S of deterministic points in \mathbb{R}^d (which may not be a subset of \mathcal{P}) such that $\omega(u, \mathcal{P}) = \omega(u, S)$ for all $u \in \mathbb{R}^d$.*

Proof. By the definition of the expected directional width of \mathcal{P} , we have that

$$\omega(\mathcal{P}, u) = \mathbb{E}_{P \sim \mathcal{P}}[\omega(P, u)] = \sum_{P \sim \mathcal{P}} \Pr[P] \left(f(P, u) + f(P, -u) \right).$$

Consider the Minkowski sum $M = M(\mathcal{P}) := \sum_{P \sim \mathcal{P}} \Pr[P] \text{ConvH}(P)$, where $\text{ConvH}(P)$ is the convex hull of P (including the interior). It is well known that the Minkowski sum of a set of convex sets is also convex. Moreover, it also holds that for all $u \in \mathbb{R}^d$ (see e.g., [60]) $f(M, u) = \sum_{P \sim \mathcal{P}} \Pr[P] f(P, u)$. Hence, $\omega(\mathcal{P}, u) = \omega(M, u)$ for all $u \in \mathbb{R}^d$. \square

By the result in [7], we know that for any convex body in \mathbb{R}^d , there exists an ε -kernel of size $O(\varepsilon^{-(d-1)/2})$. Combining with Lemma 10, we can immediately obtain the following corollary, which is the first half of Theorem 4.

Corollary 11. *For any $\varepsilon > 0$, there exists an ε -EXP-KERNEL of size $O(\varepsilon^{-(d-1)/2})$.*

Recall that in Lemma 10, the Minkowski sum $M = \sum_{P \sim \mathcal{P}} \Pr[P] \text{ConvH}(P)$. Since M is the Minkowski sum of exponential many convex polytopes, so M is also a convex polytope. At first sight, the complexity of M (i.e., number of vertices) could be exponential. However, as we will show shortly, the complexity of M is in fact polynomial.

We need some notations first. For each pair (r, w) of points in \mathcal{P} consider the hyperplane $H_{r,w}$ that passes through the origin and is orthogonal to the line connecting r and w . We call these $\binom{n}{2}$ hyperplanes *the separating hyperplanes induced by \mathcal{P}* and use Γ to denote the set. Each such hyperplane divides \mathbb{R}^d into 2 halfspaces. For all vectors $u \in \mathbb{R}^d$ in each halfspace, the order of $\langle r, u \rangle$ and $\langle w, u \rangle$ is the same (i.e., we have $r \succ_u w$ in one halfspace and $w \succ_u r$ in the other). Those hyperplanes in Γ pass through the origin and thus partition \mathbb{R}^d into d -dimensional polyhedra cones.² We denote this *arrangement* as $\mathbb{A}(\Gamma)$.

Consider an arbitrary cone $C \in \mathbb{A}(\Gamma)$. Let $\text{int } C$ denote the interior of C . We can see that for all vectors $u \in \text{int } C$, the canonical order of \mathcal{P} with respect to u is the same (since all vector $u \in \text{int } C$ lie in the same set of halfspaces). We use $|M|$ to denote the complexity of M , i.e., the number of vertices in $\text{ConvH}(M)$.

Lemma 12. *Assuming the existential model and $p_v \in (0, 1)$ for all $v \in \mathcal{P}$, the complexity of M is the same as the cardinality of $\mathbb{A}(\Gamma)$, i.e., $|M| = |\mathbb{A}(\Gamma)|$. Moreover, each cone $C \in \mathbb{A}(\Gamma)$ corresponds to exactly one vertex v of $\text{ConvH}(M)$ in the following sense: the gradient $\nabla f(M, u) = v$ for all $u \in \text{int } C$ (note that here v should be understood as a vector).*

² We ignore the lower dimensional cells in the arrangement.

Proof. We have shown that M is a convex polytope. We first note that the support function uniquely defines a convex body (see e.g., [60]). We need the following well known fact in convex geometry (see e.g., [34]): For any convex polytope M , \mathbb{R}^d can be divided into exactly $|M|$ polyhedra cones (of dimension d , ignoring the boundaries), such that each such cone C_v corresponds to a vertex v of M , and for each vector $u \in C_v$, it holds $f(M, u) = \langle u, v \rangle$ (i.e., the maximum of $f(M, u) = \max_{v' \in M} \langle u, v' \rangle$ is achieved by v for all $u \in C_v$).³ See Figure 1 for an example in \mathbb{R}^2 . Hence, for each $u \in \text{int } C_v$ the gradient of the support function (as a function of u) is exactly v :

$$\nabla f(M, u) = \left\{ \frac{\partial f(M, u)}{\partial u_j} \right\}_{j \in [d]} = \left\{ \frac{\partial \langle u, v \rangle}{\partial u_j} \right\}_{j \in [d]} = \left\{ \frac{\partial \sum_{j \in [d]} v_j u_j}{\partial u_j} \right\}_{j \in [d]} = v, \quad (2)$$

where u_j is the j th coordinate of u . With a bit abuse of notation, we denote the set of cones defined above by $\mathbb{A}(M)$.

Now, consider a cone $C \in \mathbb{A}(\Gamma)$. We show that for all $u \in \text{int } C$, $\nabla f(M, u)$ is a distinct constant vector independent of u . In fact, we know that $f(M, u) = f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u) \langle v, u \rangle$, where $\text{Pr}^R(v, u) = \prod_{v' \succ_u v} (1 - p_{v'}) p_v$. For all $u \in \text{int } C$, the $\text{Pr}^R(v, u)$ value is the same since the value only depends on the canonical order with respect to u , which is the same for all $u \in C$. Hence, we can get that for all $u \in \text{int } C$,

$$\nabla f(M, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u) v, \quad (3)$$

which is a constant independent of u . We prove the lemma by showing that the gradient $\nabla f(M, u)$ must be different for two adjacent cones C_1, C_2 (separated by some hyperplane in Γ) in $\mathbb{A}(\Gamma)$. Suppose $u_1 \in \text{int } C_1$ and $u_2 \in \text{int } C_2$. Consider the canonical orders O_1 and O_2 of \mathcal{P} with respect to u_1 and u_2 respectively. Since C_1 and C_2 are adjacently, O_1 and O_2 only differ by one swap of adjacent vertices. W.l.o.g., assume that $O_1 = \{v_1, \dots, v_i, v_{i+1}, \dots, v_n\}$ and $O_2 = \{v_1, \dots, v_{i+1}, v_i, \dots, v_n\}$. Using (3), we can get that

$$\begin{aligned} \nabla f(M, u_1) - \nabla f(M, u_2) &= \text{Pr}^R(v_i, u_1) v_i + \text{Pr}^R(v_{i+1}, u_1) v_{i+1} - \text{Pr}^R(v_i, u_2) v_i - \text{Pr}^R(v_{i+1}, u_2) v_{i+1} \\ &= D \cdot (p_{v_i} v_i + (1 - p_{v_i}) p_{v_{i+1}} v_{i+1} - p_{v_{i+1}} v_{i+1} - (1 - p_{v_{i+1}}) p_{v_i} v_i) \\ &= D \cdot p_{v_i} p_{v_{i+1}} (v_i - v_{i+1}) \neq 0 \end{aligned}$$

where $D = \prod_{j=1}^{i-1} (1 - p_{v_j}) \neq 0$.

In summary, we have shown in the first paragraph that $\nabla f(M, u)$ is piecewise constant, with a distinct constant in each cone in $\mathbb{A}(M)$. The same also holds for $\mathbb{A}(\Gamma)$. This is only possible if $\mathbb{A}(\Gamma)$ (thinking as a partition of \mathbb{R}^d) partitions \mathbb{R}^d exactly the same way as $\mathbb{A}(M)$ does. Hence, we have $\mathbb{A}(\Gamma) = \mathbb{A}(M)$ and the lemma follows immediately. \square

Since $O(n^2)$ hyperplanes passing through the origin can divide \mathbb{R}^d into at most $O(\binom{n^2}{d-1})$ d -dimensional polyhedra cones (see e.g., [11]), we immediately obtain the following corollary.

³ One intuitive way to see this is as follows: The support function for a polytope is just the upper envelope of a finite set of linear functions, thus a piecewise linear function, and the domain of each piece is a polyhedra cone.

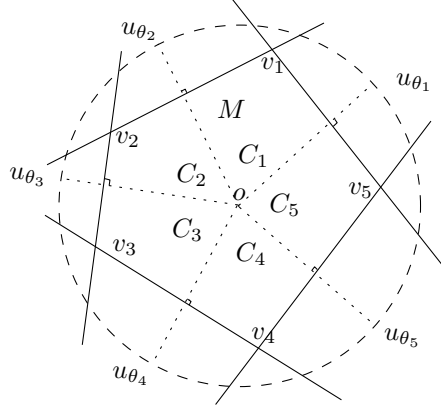


Figure 1: The figure depicts a pentagon M in \mathbb{R}^2 to illustrate some intuitive facts in convex geometry. (1) The plane can be divided into 5 cones C_1, \dots, C_5 , by 5 angles $\theta_1, \dots, \theta_5$. u_{θ_i} is the unit vector corresponding to angle θ_i . Each cone C_i corresponds to a vertex v_i and for any direction $u \in C_i$, $f(M, u) = \langle u, v_i \rangle$ and the vector $\nabla f(M, u)$ is v_i . (2) Each direction θ_i is perpendicular to an edge of M . $M = \cap_{i=1}^5 H_i$ where H_i is the supporting halfplane with normal vector u_{θ_i} .

Corollary 13. *It holds that $|M| \leq O(\binom{n^2}{d-1}) = O(n^{2d-2})$.*

The proof of Lemma 12 can be easily made constructive. We only need to compute the set Γ of all $O(n^2)$ hyperplanes and the arrangement $\mathbb{A}(\Gamma)$ in $O(n^{2d-2})$ time (see e.g., [11, 28]). Given each cone $C \in \mathbb{A}(\Gamma)$, we can calculate $\nabla f(M, u)$ for any $u \in C$, which gives exactly one vertex of M by (2), in $O(n \log n)$ time using the algorithm described in Lemma 43.

Theorem 14. *In \mathbb{R}^d for constant d , the polytope M which defines $f(\mathcal{P}, u)$ for any direction u can be described with $O(n^{2d-2})$ vertices in \mathbb{R}^d , and can be computed in $O(n^{2d-1} \log n)$ time. In \mathbb{R}^2 , the runtime can be improved to $O(n^2 \log n)$.*

The improved running time in \mathbb{R}^2 is derived in Lemma 44 by carefully constructing each vertex of M in $O(1)$ time using its neighboring vertex. The extra $O(\log n)$ is needed to sort the vertices of M to determine neighbors.

2.1 A Nearly Linear Time Algorithm for Constructing ε -exp-kernels

Now, we prove the main algorithmic result Theorem 4 of this section: we can find an ε -EXP-KERNEL in nearly linear time. If we already have the Minkowski sum M , we can directly use the algorithm in [7] to find an ε -kernel for M . However, constructing M explicitly takes $O(n^{2d-1} \log n)$ time according to Theorem 14 and this cannot be improved in general as the complexity of M is $O(n^{2d-2})$. Therefore, in order to achieve a nearly linear time coresset construction, we can not compute M explicitly. For ease of description, we first consider existential uncertainty model. The details for locational uncertainty model can be found in Appendix A.

Theorem 15. (second half of Theorem 4, for existential model) \mathcal{P} is a set of n uncertain points in \mathbb{R}^d with existential uncertainty. An ε -EXP-KERNEL of size $O(\varepsilon^{-(d-1)/2})$ for \mathcal{P} can be constructed in $O(\varepsilon^{-(d-1)}n \log n)$ time.

The following simple lemma provides an efficient procedure for finding the extreme vertex in M along any give direction, and is useful in several places later as well.

Lemma 16. Given any direction $u \in \mathbb{R}^d$, we can find in $O(n \log n)$ time a vertex $v^* \in M$, at which $\langle v, u \rangle$ is maximized, over all $v \in M$.

Proof. Fix an arbitrary direction $u \in \mathbb{R}^d$. From the proof of Lemma 12 (in particular (2)), we know that the vertex $v^* \in M$ that maximizes $\langle v, u \rangle$ can be computed by $v^* = \nabla f(M, u)$. Using (3), $\nabla f(M, u)$ can be easily computed in $O(n \log n)$ time (see Lemma 43 for the details). \square

Next, we need to find an affine transform T such that the convex polytope $M' = T(M)$ is α -fat for some constant α . We recall that a set P of points is α -fat, for some constant $\alpha \leq 1$, if there exists a point $x \in \mathbb{R}^d$, and a unit hypercube $\overline{\mathbb{C}}$ centered at x such that $\alpha \overline{\mathbb{C}} \subset \text{ConvH}(P) \subset \overline{\mathbb{C}}$. According to Chapter 22 in [39], in order to construct such T , it suffices to identify two points in M such that their distance is a constant approximation of the diameter of M . The following lemma (proven in Appendix A) shows this can be done without computing M explicitly.

Lemma 17. We find an affine transform T in $O(2^{O(d)}n \log n)$ time, such that the convex polytope $M' = T(M)$ is α -fat for some constant α (α may depend on d).

After obtaining T , we apply T to \mathcal{P} in linear time. Notice that $M' = T(M(\mathcal{P})) = M(T(\mathcal{P}))$. Therefore, Lemma 16 also holds for M' (i.e., we can search over M' the maximum vertex in any given direction in $O(n \log n)$ time).

Let $\delta = O(\varepsilon \alpha / d)$. We compute a set \mathcal{I} of $O(\delta^{-(d-1)}) = O(\varepsilon^{-(d-1)})$ points on the unit sphere \mathbb{S}^{d-1} such that for any point $v \in \mathbb{S}^{d-1}$, there is a point $u \in \mathcal{I}$ such that $\|u - v\| \leq \delta$ (see e.g., [10, 16]). For each u in \mathcal{I} , we include $-u$ in \mathcal{I} as well. For each direction $u \in \mathcal{I}$, we compute $x(u) = \arg \max_{x \in M'} \langle x, u \rangle$. Based on the previous discussion, all $\{x(u)\}_{u \in \mathcal{I}}$ can be computed in $O(\delta^{-(d-1)}n \log n) = O(\varepsilon^{-(d-1)}n \log n)$ time.

Lemma 18. $S = \{x(u)\}_{u \in \mathcal{I}}$ is an ε -kernel for M' .⁴

Finally, we can then run existing ε -kernel algorithms [17, 65] in $O(|S|)$ time to further reduce the size of S to $O(\varepsilon^{-(d-1)/2})$, which finishes the proof of Theorem 15. Lemma 12 and Theorem 15 hold for locational uncertainty models as well. The details can be found in Appendix A.

⁴ This is a folklore result. A proof of the 2D case can be found in [21]. The general case is a straightforward extension and we provide a proof in Appendix A for completeness.

2.2 ε -exp-kernel Under the Subset Constraint

First, we show that under the subset constraint (i.e., the ε -EXP-KERNEL is required to be a subset of the original point set, with the same probability distribution for each chosen point), there exists no ε -EXP-KERNEL with small size in general.⁵

Lemma 19. *For some constant $\varepsilon > 0$, there exist a set \mathcal{P} of stochastic points such that no $o(n)$ size ε -EXP-KERNEL exists for \mathcal{P} under the subset constraint (for both locational model and existential model).*

Proof. To see this in the existential uncertainty model, simply consider n points, each with existence probability $1/n$. $n/2$ of them co-locate at the origin and the other $n/2$ of them co-locate at $x = 1$. It is not hard to see that the expected length of the diameter is $\Omega(1)$ but the expected length of the diameter of any $o(n)$ size subset is only $o(1)$ (with high probability, no point would even appear).

The case for the locational uncertainty model is as simple. Again, consider n points. For each point, with probability $1/n$, it appears at $x = 1$. Otherwise, its position is the origin (with probability $1 - 1/n$). It is not hard to see that the expected length of the diameter of the original point set is $\Omega(1)$, while that of any $o(n)$ size subset is only $o(1)$ (with high probability, no point would realize at $x = 1$). \square

In light of the above negative result, we make the following β -assumption: we assume each possible location realizes a point with probability at least β , for a constant $\beta > 0$. The proof of the following theorem can be found in Appendix A.

Theorem 20. *Under the β -assumption, in the existential uncertainty model, there is an ε -EXP-KERNEL in \mathbb{R}^d of size $O(\beta^{-(d-1)}\varepsilon^{-(d-1)/2}\log(1/\varepsilon))$ that satisfies the subset constraint.*

3 ε -Kernels for Probability Distributions of Width

Recall \mathcal{S} is an (ε, τ) -QUANT-KERNEL if for all $x \geq 0$, $\Pr_{P \sim \mathcal{P}}[\omega(P, u) \leq (1 - \varepsilon)x] - \tau \leq \Pr_{P \sim \mathcal{S}}[\omega(S, u) \leq x] \leq \Pr_{P \sim \mathcal{P}}[\omega(P, u) \leq (1 + \varepsilon)x] + \tau$. For ease of notation, we sometimes write $\Pr[\omega(\mathcal{P}, u) \leq t]$ to denote $\Pr_{P \sim \mathcal{P}}[\omega(P, u) \leq t]$, and abbreviate the above as $\Pr[\omega(S, u) \leq x] \in \Pr[\omega(\mathcal{P}, u) \leq (1 \pm \varepsilon)x] \pm \tau$. We first provide a simple linear time algorithm for constructing an (ε, τ) -QUANT-KERNEL for both existential and locational models, in Section 3.1. The points in the constructed kernel are not independent. Then, for existential models, we provide a nearly linear time (ε, τ) -QUANT-KERNEL construction where all stochastic points in the kernel are independent in Section 3.2.

⁵If we require the ε -EXP-KERNEL to be a subset of the original point set, but with possibly different probability for each chosen point, we do not know whether there always exists an ε -EXP-KERNEL with small size.

3.1 A Simple (ε, τ) -quant-kernel Construction

In this section, we show a linear time algorithm for constructing an (ε, τ) -QUANT-KERNEL for any stochastic model if we can sample a realization from the model in linear time (which is true for both locational and existential uncertainty models).

Algorithm: Let $N = O(\tau^{-2}\varepsilon^{-(d-1)} \log \frac{1}{\varepsilon})$. We sample N independent realizations from the stochastic model. Let \mathcal{H}_i be the convex hull of the present points in the i th realization. For \mathcal{H}_i , we use the algorithm in [7] to find a deterministic ε -kernel \mathcal{E}_i of size $O(\varepsilon^{-(d-1)/2})$. Our (ε, τ) -QUANT-KERNEL \mathcal{S} is the following simple stochastic model: with probability $1/N$, all points in \mathcal{E}_i are present. Hence, \mathcal{S} consists of $O(\tau^{-2}\varepsilon^{-3(d-1)/2} \log \frac{1}{\varepsilon})$ points (two such points either co-exist or are mutually exclusive). Hence, for any direction u , $\Pr[\omega(\mathcal{S}, u) \leq t] = \frac{1}{N} \sum_{i=1}^N \mathbb{I}(\omega(\mathcal{E}_i, u) \leq t)$, where $\mathbb{I}(\cdot)$ is the indicator function.

For a realization $P \sim \mathcal{P}$, we use $\mathcal{E}(P)$ to denote the deterministic ε -kernel for P . So, $\mathcal{E}(P)$ is a random set of points, and we can think of $\mathcal{E}_1, \dots, \mathcal{E}_N$ as samples from the random set. Now, we show \mathcal{S} is indeed an (ε, τ) -QUANT-KERNEL. We start with the following simple observation.

Observation 21. *For any $t \geq 0$ and any direction u , we have that*

$$\Pr[\omega(\mathcal{P}, u) \leq t] \leq \Pr_{P \sim \mathcal{P}}[\omega(\mathcal{E}(P), u) \leq t] \leq \Pr[\omega(\mathcal{P}, u) \leq (1 + \varepsilon)t].$$

Proof. For any realization P of \mathcal{P} , we have $\frac{1}{1+\varepsilon}\omega(P, u) \leq \omega(\mathcal{E}(P), u) \leq \omega(P, u)$. The observation follows by combining all realizations. \square

We only need to show that \mathcal{S} is an (ε, τ) -QUANT-KERNEL for $\mathcal{E}(P)$. We need the following two theorems.

Theorem 22. *(Theorem 5.22 in [39]) (VC-dimension) Let $S_1 = (X, \mathcal{R}^1), \dots, S_k = (X, \mathcal{R}^k)$ be range spaces with VC-dimension $\delta_1, \dots, \delta_k$, respectively. Next, let $f(r_1, \dots, r_k)$ be a function that maps any k -tuple of sets $r_1 \in \mathcal{R}^1, \dots, r_k \in \mathcal{R}^k$ into a subset of X . Consider the range set*

$$\mathcal{R}' = \{f(r_1, \dots, r_k) \mid r_1 \in \mathcal{R}^1, \dots, r_k \in \mathcal{R}^k\}$$

and the associated range space (X, \mathcal{R}') . Then, the VC-dimension of (X, \mathcal{R}') is bounded by $O(k\delta \log k)$, where $\delta = \max_i \delta_i$.

Suppose (X, \mathcal{R}) is a range space and μ is a probability measure over X . We say a subset $C \subset X$ is an ε -approximation of the range space if for any range $R \in \mathcal{R}$, we have $|\mu_C(R) - \mu(R)| \leq \varepsilon$, where $\mu_C(R) = |C \cap R|/|C|$. We need the following celebrated uniform convergence result, first established by Vapnik and Chervonenkis [63].

Theorem 23. *(See Theorem 4.9 in [12]) Suppose (X, \mathcal{R}) is any range space with VC-dimension at most V , where $|X|$ is finite and μ is a probability measure defined over X . For any $\varepsilon, \delta > 0$, a random subset $C \subseteq X$ (according to μ) of cardinality $s = O(\varepsilon^{-2}(V + \log(1/\delta)))$ is an ε -approximation for X with probability $1 - \delta$.*

Now, we are ready to prove the main lemma in this section.

Lemma 24. *Let $N = O(\tau^{-2}\varepsilon^{-(d-1)}\log(1/\varepsilon))$. For any $t \geq 0$ and any direction u ,*

$$\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr_{P \sim \mathcal{P}}[\omega(\mathcal{E}(P), u) \leq t] \pm \tau.$$

Proof. Let $L = O(\varepsilon^{-(d-1)/2})$. We first note that $\mathcal{E}(P)$ has at most n^L possible realizations since each ε -kernel is of size at most L . We first build a mapping g that maps each realization $\mathcal{E}(P)$ to a point in \mathbb{R}^{dL} , as follows: Consider a realization P of \mathcal{P} . Suppose $\mathcal{E}(P) = \{(x_1^1, \dots, x_d^1), \dots, (x_1^L, \dots, x_d^L)\}$ (if $|\mathcal{E}(P)| < L$, we pad it with $(0, \dots, 0)$). We let

$$g(\mathcal{E}(P)) = (x_1^1, \dots, x_d^1, \dots, x_1^L, \dots, x_d^L) \in \mathbb{R}^{dL}.$$

For any $t \geq 0$ and any direction $u \in \mathbb{R}^d$, note that $\omega(\mathcal{E}(P), u) \geq t$ holds if and only if there exists some $1 \leq i, j \leq |\mathcal{E}(P)|, i \neq j$ satisfies that $\sum_{k=1}^d (x_k^i - x_k^j)u_k \geq t$, which is equivalent to saying that point $g(\mathcal{E}(P))$ is in the union of those $O(|\mathcal{E}(P)|^2)$ halfspaces (for each i, j , we have one such halfspace).

Let X be the image set of g . Let $(X, \mathcal{R}^{i,j})$ ($1 \leq i, j \leq L, i \neq j$) be a range space, where $\mathcal{R}^{i,j}$ is the set of halfspaces $\{u = (u_1, \dots, u_d) \in \mathbb{R}^d \mid \sum_{k=1}^d (x_k^i - x_k^j)u_k \geq t\}$. Let $\mathcal{R}' = \{\cup \mathcal{R}^{i,j} \mid r_{i,j} \in \mathcal{R}^{i,j}, i, j \in [L]\}$. Note that each $(X, \mathcal{R}^{i,j})$ has VC-dimension $d+1$. By Theorem 22, we can see that the VC-dimension of (X, \mathcal{R}') is bounded by $O((d+1)L^2 \lg L^2) = O(\varepsilon^{-(d-1)}\log(1/\varepsilon))$. Notice that $\mathcal{S} = \{\mathcal{E}_1, \dots, \mathcal{E}_N\}$ is a collection of samples from $\mathcal{E}(P)$. Hence, by Theorem 23, for any t and any direction u , we have that $\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr_{P \sim \mathcal{P}}[\omega(\mathcal{E}(P), u) \leq t] \pm \tau$. \square

Combining Observation 21 and Lemma 24, we obtain the following theorem.

Theorem 25. *Let $N = O(\tau^{-2}\varepsilon^{-(d-1)}\log(1/\varepsilon))$. For any $t \geq 0$ and any direction u , we have that*

$$\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr[\omega(\mathcal{P}, u) \leq (1 \pm \varepsilon)t] \pm \tau.$$

Running time: In each sample, the size of an ε -kernel \mathcal{K}_i is at most $O(\varepsilon^{-(d-1)/2})$. Note that we can compute \mathcal{K}_i in $O(n + \varepsilon^{-(d-3/2)})$ time [17, 65]. We take $O(\tau^{-2}\varepsilon^{-(d-1)}\log(1/\varepsilon))$ samples in total. So the overall running time is $O(n\tau^{-2}\varepsilon^{-(d-1)}\log(1/\varepsilon) + \text{poly}(\frac{1}{\varepsilon\tau})) = \tilde{O}(n\tau^{-2}\varepsilon^{-(d-1)})$. In summary, we obtain our main result for (ε, τ) -QUANT-KERNEL in this subsection.

Theorem 5. *(restated) An (ε, τ) -QUANT-KERNEL of size $\tilde{O}(\tau^{-2}\varepsilon^{-3(d-1)/2})$ can be constructed in $\tilde{O}(n\tau^{-2}\varepsilon^{-(d-1)})$ time, under both existential and locational uncertainty models.*

3.2 Improved (ε, τ) -quant-kernel for Existential Models

In this section, we show an (ε, τ) -QUANT-KERNEL \mathcal{S} can be constructed in nearly linear time for the existential model, and all points in \mathcal{S} are independent of each other. The size bound $\tilde{O}(\tau^{-2}\varepsilon^{-(d-1)})$ (see Theorem 6) is better than that in Theorem 5 for the general case, and the independence property may be useful in certain applications. Moreover, some of the insights developed in this section may be of independent interest (e.g., the connection to

Tukey depth). Due to the independence requirement, the construction is somewhat more involved. For ease of the description, we assume the Euclidean plane first. All results can be easily extended to \mathbb{R}^d . We also assume that all probability values are strictly between 0 and 1 and $0 < \varepsilon, \tau \leq 1/2$ is a fixed constant.

Let $\lambda(\mathcal{P}) = \sum_{v \in \mathcal{P}} (-\ln(1 - p_v))$. In the following, we present two algorithms. The first algorithm works for any $\lambda(\mathcal{P})$ and produces an (ε, τ) -QUANT-KERNEL \mathcal{S} whose size depends on $\lambda(\mathcal{P})$. In Section 3.2.2, we present the second algorithm that only works for $\lambda(\mathcal{P}) \geq 3 \ln(2/\tau)$ but produces an (ε, τ) -QUANT-KERNEL \mathcal{S} with a constant size (the constant only depends on ε, τ and δ). Thus, we can get a constant size (ε, τ) -QUANT-KERNEL by running the first algorithm when $\lambda(\mathcal{P}) \leq 3 \ln(2/\tau)$ and running the second algorithm otherwise.

3.2.1 Algorithm 1: For Any $\lambda(\mathcal{P})$

In this section, we present the first algorithm which works for any $\lambda(\mathcal{P})$. We can think of each point v associated with a Bernoulli random variable X_v that takes value 1 with probability p_v and 0 otherwise. Now, we replace the Bernoulli random variable X_v by a Poisson distributed random variable \tilde{X}_v with parameter $\lambda_v = -\ln(1 - p_v)$ (denoted by $\text{Pois}(\lambda_v)$), i.e., $\Pr[\tilde{X}_v = k] = \frac{1}{k!} \lambda_v^k e^{-\lambda_v}$, for $k = 0, 1, 2, \dots$. Here, $\tilde{X}_v = k$ means that there are k realized points located at the position of v . We call the new instance *the Poissonized instance corresponding to \mathcal{P}* . We can check that $\Pr[\tilde{X}_v = 0] = e^{-\lambda_v} = 1 - p_v = \Pr[X_v = 0]$. Also note that co-locating points do not affect any directional width, so the Poissonized instance is essentially equivalent to the original instance for our problem.

The construction of the (ε, τ) -QUANT-KERNEL \mathcal{S} is as follows: Let \mathfrak{A} be the probability measure over all points in \mathcal{P} defined by $\mathfrak{A}(\{v\}) = \lambda_v/\lambda$ for every $v \in \mathcal{P}$, where $\lambda := \lambda(\mathcal{P}) = \sum_{v \in \mathcal{P}} \lambda_v$. Let τ_1 be a small positive constant to be fixed later. We take $N = \tilde{O}(\tau_1^{-2})$ independent samples from \mathfrak{A} (we allow more than one point to be co-located at the same position), and let \mathfrak{B} be the empirical measure, i.e., each sample point having probability $1/N$. The coreset \mathcal{S} consists of the N sample points in \mathfrak{B} , each with the same existential probability $1 - \exp(-\lambda/N)$. A useful alternative view of \mathcal{S} is to think of each point associated with a random variable Y_v following distribution $\text{Pois}(\lambda/N)$ (i.e., the Poissonized instance corresponding to \mathcal{S}). This finishes the description of the construction.

Now, we start the analysis. Our goal is to show that \mathcal{S} is indeed an (ε, τ) -QUANT-KERNEL. The following theorem is a special case of Theorem 23 (specialized to the range space consisting of all halfplanes), which shows that the empirical measure \mathfrak{B} is close to the original measure \mathfrak{A} with respect to all half spaces.

Theorem 26. [12, 50] *We denote the set of all halfplanes by \mathbb{H} . With probability $1 - \delta$, the empirical measure \mathfrak{B} (defined by $N = O(\tau_1^{-2} \log(1/\delta))$ independent samples) satisfies the following:*

$$\sup_{H \in \mathbb{H}} |\mathfrak{A}(H) - \mathfrak{B}(H)| \leq \tau_1.$$

From now on, we assume that \mathfrak{B} satisfies the statement of Theorem 26. We first observe a simple but useful lemma, which is a consequence of Theorem 26. For a halfplane H , we use $H \models 0$ to denote the event that no point is realized in H .

Lemma 27. *With probability $1 - \delta$, for any halfplane $H \in \mathbb{H}$, we have that*

$$\Pr_{\mathcal{S}}[H \models 0] \in (1 \pm O(\lambda\tau_1))\Pr_{\mathcal{P}}[H \models 0].$$

Proof. Fix an arbitrary halfplane $H \in \mathbb{H}$. Consider the Poissonized instance corresponding to \mathcal{P} . We first observe that $\Pr_{\mathcal{P}}[H \models 0] = \Pr_{\mathcal{P}}[\sum_{v \in \mathcal{P} \cap H} X_v = 0]$. Since X_v follows distribution $\text{Pois}(\lambda_v)$, $\sum_{v \in \mathcal{P} \cap H} X_v$ follows Poisson distribution $\text{Pois}(\sum_{v \in \mathcal{P} \cap H} \lambda_v)$. Similarly, we have that $\Pr_{\mathcal{S}}[H \models 0] = \Pr_{\mathcal{S}}[\sum_{v \in \mathcal{S} \cap H} Y_v = 0]$ since $\sum_{v \in \mathcal{S} \cap H} Y_v$ follows $\text{Pois}(\sum_{v \in \mathcal{S} \cap H} \lambda/N)$. Hence, we can see the following:

$$\begin{aligned} \Pr_{\mathcal{P}}[H \models 0] &= \exp\left(-\sum_{v \in \mathcal{P} \cap H} \lambda_v\right) = \exp\left(-\lambda \mathfrak{A}(H)\right) \\ &\in \exp\left(-\lambda(\mathfrak{B}(H) \pm \tau_1)\right) = \exp\left(-\sum_{v \in \mathcal{S} \cap H} \frac{\lambda}{N} \pm \tau_1 \lambda\right) \\ &\in (1 \pm O(\lambda\tau_1)) \exp\left(-\sum_{v \in \mathcal{S} \cap H} \frac{\lambda}{N}\right) = (1 \pm O(\lambda\tau_1))\Pr_{\mathcal{S}}[H \models 0]. \end{aligned}$$

The first inequality follows from Theorem 26 and the second is due to the fact that $e^{-\varepsilon} \geq 1 - \varepsilon$ and $e^{\varepsilon} \leq 1 + (e - 1)\varepsilon$ for any $0 < \varepsilon < 1$. \square

For two real-valued random variables X, Y , we define the Kolmogorov distance $d_K(X, Y)$ between X and Y to be $d_K(X, Y) = \sup_{t \in \mathbb{R}} |\Pr[X \leq t] - \Pr[Y \leq t]|$. We also need the following simple lemma.

Lemma 28. *Suppose we have four independent random variables X, X', Y and Y' such that $d_K(X, X') \leq \varepsilon$ and $d_K(Y, Y') \leq \varepsilon$ for some $\varepsilon \geq 0$. Then, $d_K(X + Y, X' + Y') \leq 2\varepsilon$.*

Proof. We need the following useful elementary fact about Kolmogorov distance: Let X, Y, Z be real-valued random variables such that X is independent of Y and independent of Z . Then we have that $d_K(X + Y, X + Z) \leq d_K(Y, Z)$. The rest of the proof is straightforward: $d_K(X + Y, X' + Y') \leq d_K(X + Y, X + Y') + d_K(X + Y', X' + Y') \leq 2\varepsilon$. The first inequality is the triangle inequality. \square

Now, we are ready to show that \mathcal{S} is really an (ε, τ) -QUANT-KERNEL. We note that in this subsection our bound is stronger than (1) in that we do not need to relax the length threshold. We first prove the theorem under a simplified assumption: we assume that there is a point $v^* \in \mathbb{R}^2$ (not necessarily an input point), which we call the special point, that lies in the convex hull of \mathcal{P} with probability at least $1 - \delta/2$. With the assumption, the proof is much simpler but still instructive as the analysis in Section 3.2.2 is an extension of this proof. The general case is proved in Theorem 30 and the proof is more technical and the size bound is slightly worse.

Theorem 29. *Assume that there is a special point $v^* \in \mathbb{R}^2$ that lies in the convex hull of \mathcal{P} with probability at least $1 - \delta/2$. The parameters of the algorithm are set as*

$$\tau_1 = O\left(\frac{\tau}{\lambda}\right) \quad \text{and} \quad N = O\left(\frac{1}{\tau_1^2} \log \frac{1}{\delta}\right) = O\left(\frac{\lambda^2}{\tau^2} \log \frac{1}{\delta}\right).$$

With probability at least $1 - \delta$, for any $t \geq 0$ and any direction u , we have that

$$\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr[\omega(\mathcal{P}, u) \leq t] \pm \tau. \quad (4)$$

Proof. We first condition on the event that v^* is in the convex hull of all realized points (which happens with probability at least $1 - \delta/2$). The remainder needs to hold with probability at least $1 - \delta/2$. Under the condition, we can pretend that v^* is a deterministic point in the original point set (this does not affect any directional width as v^* is in the convex hull).

Fix an arbitrary direction u (w.l.o.g., say it is the x -axis). Rename all points as v_1, v_2, \dots, v_n according to the increasing order of their projections to u . Suppose v^* is renamed as v_k . Let the random variable L be the directional width of $\{v_1, \dots, v_k\}$ with respect to u and R be the directional width of $\{v_k, \dots, v_n\}$ with respect to u . Since v^* is assumed to be within the left and right extents, we can easily see that $\omega(\mathcal{P}, u) = L + R$. Similarly, we define L' (R' resp.) to be the directional width of all points in \mathcal{S} to the left (right resp.) of v^* . Since the convex hull of \mathcal{S} contains v^* , we can also see that $\omega(\mathcal{S}, u) = L' + R'$. By Lemma 27, we know that $d_K(L, L') \leq O(\lambda\tau_1)$ and $d_K(R, R') \leq O(\lambda\tau_1)$. By Lemma 28, we have that $d_K(\omega(\mathcal{S}, u), \omega(\mathcal{P}, u)) \leq O(\lambda\tau_1)$. Let $\tau_1 = O(\tau/\lambda)$, the theorem follows. \square

Now, we prove the theorem in the general case, where the main difficulty comes from the fact that we can not separate the width into two independent parts L and R . The proof is somewhat technical and can be found in Appendix B.

Theorem 30. *Let $\tau_1 = O(\frac{\tau}{\max\{\lambda, \lambda^2\}})$ and $N = O(\frac{1}{\tau_1^2} \log \frac{1}{\delta}) = O(\frac{\max\{\lambda^2, \lambda^4\}}{\tau^2} \log \frac{1}{\delta})$. With probability at least $1 - \delta$, for any $t \geq 0$ and any direction u , we have that $\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr[\omega(\mathcal{P}, u) \leq t] \pm \tau$.*

3.2.2 Algorithm 2: For $\lambda(\mathcal{P}) > 3 \ln(2/\tau)$

In the second algorithm, we assume that $\lambda(\mathcal{P}) = \sum_{v \in \mathcal{P}} \lambda_v > 3 \ln(2/\tau)$. When $\lambda(\mathcal{P})$ is large, we cannot directly use the sampling technique in the previous section since it requires a large number of samples. However, the condition $\lambda(\mathcal{P}) \geq 3 \ln(2/\tau)$ implies there is a nonempty convex region \mathcal{K} inside the convex hull of \mathcal{P} with high probability. Moreover, we can show the sum of λ_v values in $\bar{\mathcal{K}} = \mathbb{R}^2 \setminus \mathcal{K}$ is small. Hence, we can use the sampling technique just for $\bar{\mathcal{K}}$ and use the deterministic ε -kernel construction for \mathcal{K} .

Now, we describe the details of our algorithm. Again consider the Poissonized instance of \mathcal{P} . Imagine the following process. Fix a direction $u \in \mathbb{S}^1$.⁶ We move a sweep line ℓ_u orthogonal to u , along the direction u , to sweep through the points in \mathcal{P} . We use H_u to denote the halfplane defined by ℓ_u (with normal vector u) and \bar{H}_u denote its complement. So $\mathcal{P}(\bar{H}_u) = \mathcal{P} \cap \bar{H}_u$ is the set of points that have been swept so far. We stop the movement of ℓ_u at the first point such that $\sum_{v \in \bar{H}_u} \lambda_v \geq \ln(2/\tau)$ (ties should be broken in an arbitrary but consistent manner). One important property about \bar{H}_u is that $\Pr[\bar{H}_u \models 0] \leq \tau/2$. We

⁶Here, \mathbb{S}^1 is the surface of the unit ball in \mathbb{R}^d .

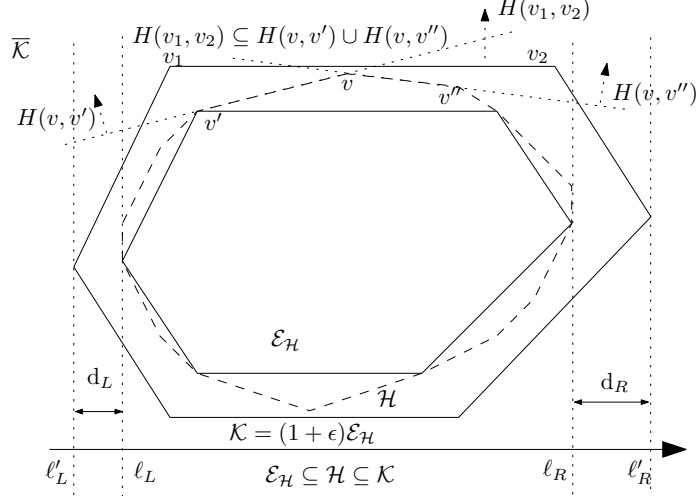


Figure 2: The construction of the (ε, τ) -QUANT-KERNEL \mathcal{S} . The dashed polygon is \mathcal{H} . The inner solid polygon is $\text{ConvH}(\mathcal{E}_{\mathcal{H}})$ and the outer one is $K = (1 + \varepsilon)\text{ConvH}(\mathcal{E}_{\mathcal{H}})$. $\bar{\mathcal{K}}$ is the set of points outside \mathcal{K} .

repeat the above process for all directions $u \in \mathbb{S}^1$ and let $\mathcal{H} = \cap_u H_u$. Since $\lambda(\mathcal{P}) > 3 \ln(2/\tau)$, by Helly's theorem, \mathcal{H} is nonempty. A careful examination of the above process reveals that \mathcal{H} is in fact a convex polytope and each edge of the polytope is defined by two points in \mathcal{P} .⁷ Moreover, \mathcal{H} is the region of points with Tukey depth at least $\ln(2/\tau)$.⁸

The construction of the (ε, τ) -QUANT-KERNEL \mathcal{S} is as follows. First, we use the algorithm in [7] to find a deterministic ε -kernel $\mathcal{E}_{\mathcal{H}}$ of size $O(\varepsilon^{-1/2})$ for \mathcal{H} . One useful property of the algorithm in [7] is that $\mathcal{E}_{\mathcal{H}}$ is a subset of the vertices of \mathcal{H} . Hence the convex polytope $\text{ConvH}(\mathcal{E}_{\mathcal{H}})$ is contained in \mathcal{H} . Since $\mathcal{E}_{\mathcal{H}}$ is an ε -kernel, $(1 + \varepsilon)\text{ConvH}(\mathcal{E}_{\mathcal{H}})$ (properly shifted) contains \mathcal{H} .⁹ Let $\mathcal{K} = (1 + \varepsilon)\text{ConvH}(\mathcal{E}_{\mathcal{H}})$ and $\bar{\mathcal{K}} = \mathcal{P} \setminus \mathcal{K}$. See Figure 2.

Now, we apply the random sampling construction over $\bar{\mathcal{K}}$. More specifically, let $\lambda := \lambda(\bar{\mathcal{K}}) = \sum_{v \in \bar{\mathcal{K}} \cap \mathcal{P}} \lambda_v$. Let \mathfrak{A} be the probability measure over $\mathcal{P} \cap \bar{\mathcal{K}}$ defined by $\mathfrak{A}(\{v\}) = \lambda_v / \lambda$ for every $v \in \mathcal{P} \cap \bar{\mathcal{K}}$. Let $\tau_1 = O(\tau/\lambda)$. We take $N = O(\tau_1^{-2} \log(1/\delta))$ independent samples from \mathfrak{A} and let \mathfrak{B} be the empirical distribution with each sample point having probability $1/N$. The (ε, τ) -QUANT-KERNEL \mathcal{S} consists of the N points in \mathfrak{B} , each with the same existential probability $1 - \exp(-\lambda/N)$, as well as all vertices of \mathcal{K} , each with probability 1. This finishes the construction of \mathcal{S} .

Now, we show that the size of \mathcal{S} is constant (only depending on ε and δ), which is an immediate corollary of the following lemma.

⁷ This also implies that we only need to do the sweep for $\binom{n}{2}$ directions. In fact, by a careful rotational sweep, we only need $O(n)$ directional sweeps.

⁸ The Tukey depth of a point $x \in \mathcal{P}$ is defined as the minimum total weight of points of \mathcal{P} contained in a closed halfspace whose bounding hyperplane passes through x .

⁹ In fact, most existing algorithms (e.g., [7]) identify a point in the interior of \mathcal{H} as origin, and compute an ε -kernel $\mathcal{E}_{\mathcal{H}}$ such that $f(\mathcal{E}_{\mathcal{H}}, u) \geq \frac{1}{1+\varepsilon} f(\mathcal{H}, u)$ for all directions u . So, $\mathcal{H} \subseteq (1 + \varepsilon)\text{ConvH}(\mathcal{E}_{\mathcal{H}})$ since $f(\mathcal{H}, u) \leq f((1 + \varepsilon)\text{ConvH}(\mathcal{E}_{\mathcal{H}}), u)$ for all directions u .

Lemma 31. $\lambda = \lambda(\overline{\mathcal{K}}) = \sum_{v \in \overline{\mathcal{K}}} \lambda_v = O\left(\frac{\ln 1/\tau}{\sqrt{\varepsilon}}\right).$

Proof. We can see that $\overline{\mathcal{K}}$ is the union of $O(\varepsilon^{-1/2})$ half-planes, each defined by a segment of \mathcal{K} . It suffices to show the sum of λ_v values in each half-plane is $O(\ln(1/\tau))$. Consider the half-plane $H(v_1, v_2)$ defined by segment (v_1, v_2) of \mathcal{K} . Suppose v is the vertex of \mathcal{H} that is closest to the line (v_1, v_2) . Let (v', v) and (v, v'') be the two edges of \mathcal{H} incident on v . Clearly, $H(v', v) \cup H(v, v'')$, the union of the two half-planes defined by (v', v) and (v, v'') , strictly contains $H(v_1, v_2)$. See Figure 2 for an illustration. Hence, $\sum_{v \in H(v_1, v_2)} \lambda_v$ is at most $2 \ln(1/\tau)$. \square

Now, we prove the main theorem in this section. The proof is an extension of Theorem 29. Here, the set \mathcal{K} plays a similar role as the special point v^* in Theorem 29. Unlike Theorem 29, we also need to relax the length threshold here, which is necessary even for deterministic points.

Theorem 32. Let $\lambda = \lambda(\overline{\mathcal{K}})$ and $\tau_1 = O(\tau/\lambda)$, and $N = O\left(\frac{1}{\tau_1^2} \log \frac{1}{\delta}\right) = O\left(\frac{\ln^2 1/\tau}{\varepsilon \tau^2} \log \frac{1}{\delta}\right)$. With probability at least $1 - \delta$, for any $t \geq 0$ and any direction u , we have that

$$\Pr\left[\omega(\mathcal{S}, u) \leq t\right] \in \Pr\left[\omega(\mathcal{P}, u) \leq (1 \pm \varepsilon)t\right] \pm \tau. \quad (5)$$

Proof. The proof is similar to Theorem 29. Fix an arbitrary direction u (w.l.o.g., say it is the x -axis). Rename all points in \mathcal{P} as v_1, v_2, \dots, v_n according to the increasing order of their x -coordinates. We use $x(v_i)$ to denote the x -coordinate of v_i . Let ℓ_L (or ℓ_R) be the vertical line that passes the leftmost endpoint of $\mathcal{E}_{\mathcal{H}}$ (or the rightmost endpoint of $\mathcal{E}_{\mathcal{H}}$). We use $x(\ell_L)$ (or $x(\ell_R)$) to denote the x coordinate of ℓ_L (or ℓ_R) and let $d(\ell_L, \ell_R) = |x(\ell_L) - x(\ell_R)|$. Suppose that v_1, \dots, v_k lie to the left of ℓ_L and v_r, \dots, v_n lie to the right of ℓ_R . Let the random variable $L = x(\ell_L) - f(\{v_1, \dots, v_k\}, -u)$ and $R = f(\{v_r, \dots, v_n\}, u) - x(\ell_R)$. Let $W = L + R + d(\ell_R, \ell_L)$. We can see that W is close to $\omega(\mathcal{P}, u)$ in the following sense. Let E denote the event that at least one point in $\{v_1, \dots, v_k\}$ is present and at least one point in $\{v_r, \dots, v_n\}$ is present. Conditioning on E , W is exactly $\omega(\mathcal{P}, u)$. Moreover, we can easily see $\Pr[E] \geq (1 - \tau/2)^2 \geq 1 - \tau$. Hence, we have

$$\begin{aligned} \Pr[W \leq t] - \tau &\leq (1 - \tau)\Pr[W \leq t] \leq \Pr[\omega(\mathcal{P}, u) \leq t \mid E]\Pr[E] \\ &\leq \Pr[\omega(\mathcal{P}, u) \leq t] = \Pr[\omega(\mathcal{P}, u) \leq t \mid E]\Pr[E] + \Pr[\omega(\mathcal{P}, u) \leq t \mid \neg E]\Pr[\neg E] \\ &\leq \Pr[W \leq t] + \tau. \end{aligned}$$

Similarly, we let ℓ'_L (or ℓ'_R) be the vertical line that passes the leftmost endpoint of \mathcal{K} (or the rightmost endpoint of \mathcal{K}). Suppose that v'_1, \dots, v'_j (points in \mathcal{S}) lie to the left of ℓ'_L and v'_s, \dots, v'_N lie to the right of ℓ'_R . We define $L' = x(\ell'_L) - f(\{v'_1, \dots, v'_j\}, -u)$ and $R' = f(\{v'_s, \dots, v'_N\}, u) - x(\ell'_R)$. We can also see that $\omega(\mathcal{S}, u) = L' + R' + d(\ell'_R, \ell'_L)$.

Let $d_L = x(\ell_L) - x(\ell'_L)$ and $d_R = x(\ell'_R) - x(\ell_R)$. Let H_t be the half-plane $\{(x, y) \mid x \leq x(\ell'_L) - t\}$. We can see that for any $t \geq 0$,

$$\begin{aligned} \Pr[L \leq t + d_L] - O(\lambda\tau_1) &= \Pr[X(H_t) = 0] - O(\lambda\tau_1) \\ &\leq \Pr[L' \leq t] = \Pr[Y(H_t) = 0] \leq \Pr[X(H_t) = 0] + O(\lambda\tau_1) \\ &= \Pr[L \leq t + d_L] + O(\lambda\tau_1), \end{aligned}$$

where the inequalities hold due to Lemma 27. Similarly, we can see that for any $t \geq 0$,

$$\Pr[R \leq t + d_R] - O(\lambda\tau_1) \leq \Pr[R' \leq t] \leq \Pr[R \leq t + d_R] + O(\lambda\tau_1).$$

Therefore, by Lemma 28, we have that for any $t > 0$,

$$\Pr[L + R \leq t + d_L + d_R] - O(\lambda\tau_1) \leq \Pr[L' + R' \leq t] \leq \Pr[L + R \leq t + d_L + d_R] + O(\lambda\tau_1).$$

Therefore, we can conclude that for any $t \geq d(\ell'_L, \ell'_R)$,

$$\begin{aligned} \omega(\Pr[\mathcal{S}, u] \leq t] &= \Pr[L' + R' + d(\ell'_L, \ell'_R) \leq t] \\ &\in \Pr[L + R + d(\ell'_L, \ell'_R) \leq t + d_L + d_R] \pm O(\lambda\tau_1) \\ &= \Pr[L + R + d(\ell_L, \ell_R) \leq t] \pm O(\lambda\tau_1) \\ &= \Pr[W \leq t] \pm O(\lambda\tau_1) \\ &= \Pr[\omega(\mathcal{P}, u) \leq t] \pm O(\lambda\tau_1 + \tau). \end{aligned}$$

Noticing that $\tau \geq \Pr[\omega(\mathcal{P}, u) \leq d(\ell_L, \ell_R)] \geq \Pr[\omega(\mathcal{P}, u) \leq (1 - \varepsilon)d(\ell'_L, \ell'_R)]$, we can obtain that, for any $t < d(\ell'_L, \ell'_R)$, $\Pr[\omega(\mathcal{S}, u) \leq t] = 0 \geq \Pr[\omega(\mathcal{P}, u) \leq (1 - \varepsilon)t] - \tau$. Moreover, it is trivially true that $\Pr[\omega(\mathcal{S}, u) \leq t] = 0 \leq \Pr[\omega(\mathcal{P}, u) \leq (1 - \varepsilon)t] + \tau$. The proof is completed. \square

Higher Dimensions: Our constructions can be easily extended to \mathbb{R}^d for any constant $d > 2$. The sampling bound (Theorem 26) still holds if the number of samples is $O(d\tau_1^{-2} \log(1/\delta)) = O(\tau_1^{-2} \log(1/\delta))$. Hence, Theorem 29 and Theorem 30 hold with the same parameters (d is hidden in the constant). In order for Algorithm 2 to work, we need $\lambda(\mathcal{P}) > (d + 1) \ln(2/\tau)$ to ensure \mathcal{H} is nonempty. Instead of constructing an ε -kernel $\mathcal{E}_{\mathcal{H}}$ with $O(\varepsilon^{-(d-1)/2})$ vertices, we construct a convex set \mathcal{K} which is the intersection of $O(\varepsilon^{-(d-1)/2})$ halfspaces and satisfies $(1 - \varepsilon)\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{K}$ (this can be done by either working with the dual, or directly using the construction implicit in [27]).

Now, we briefly sketch how to compute such \mathcal{K} using the dual approach. We first compute the dual \mathcal{H}^* of \mathcal{H} in \mathbb{R}^d . Recall the dual (also called the polar body) \mathcal{H}^* of \mathcal{H} is defined as the set $\{x \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1, y \in \mathcal{H}\}$. \mathcal{H}^* has $O(n^d)$ vertices (each corresponding to a face of \mathcal{H}). Then, compute an ε -kernel $\mathcal{E}_{\mathcal{H}^*}^*$ with $O(\varepsilon^{-(d-1)/2})$ vertices for \mathcal{H}^* . Taking the dual of $\mathcal{E}_{\mathcal{H}^*}^*$ gives the desired \mathcal{K} , which is an intersection of $O(\varepsilon^{-(d-1)/2})$ halfspaces (each corresponding to a point in $\mathcal{E}_{\mathcal{H}^*}^*$). The correctness can be easily seen by an argument through the gauge function $g(\mathcal{E}_{\mathcal{H}^*}^*, x) = \min\{\lambda \geq 0 \mid x \in \lambda \mathcal{E}_{\mathcal{H}^*}^*\}$. Since $\mathcal{E}_{\mathcal{H}^*}^* \subseteq \mathcal{H}^* \subseteq (1 + \varepsilon)\mathcal{E}_{\mathcal{H}^*}^*$, we can see that

$\frac{1}{1+\varepsilon}g(\mathcal{E}_{\mathcal{H}^*}^*, x) = g((1+\varepsilon)\mathcal{E}_{\mathcal{H}^*}^*, x) \leq g(\mathcal{H}^*, x) \leq g(\mathcal{E}_{\mathcal{H}^*}^*, x)$. The correctness follows from the duality between the gauge function and the support function, which says $g(\mathcal{E}_{\mathcal{H}^*}^*, x) = f(\mathcal{K}, x)$ and $g(\mathcal{H}^*, x) = f(\mathcal{H}, x)$ for all $x \in \mathbb{S}^{d-1}$ (see e.g., [60]).

We generalize Lemma 31 for \mathbb{R}^d by the following lemma.

Lemma 33. *There is a convex set \mathcal{K} , which is an intersection of $O(\varepsilon^{-(d-1)/2})$ halfspaces and satisfies $(1-\varepsilon)\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{K}$. Moreover, we have that $\lambda(\overline{\mathcal{K}}) = O(\varepsilon^{-(d-1)/2} \ln(1/\tau))$.*

Plugging the new bound of $\lambda(\overline{\mathcal{K}})$, we can see that it is enough to set

$$N = O(\tau^{-2}\varepsilon^{-(d-1)} \log \frac{1}{\delta} \text{polylog} \frac{1}{\tau}) = \tilde{O}(\varepsilon^{-(d-1)}\tau^{-2}).$$

Running Time: The algorithm in Section 3.2.1 takes only $O(Nn)$ time (where N is the size of the kernel, which is constant if ε , τ and λ are constant). The algorithm in Section 3.2.2 is substantially slower. The most time consuming part is the construction of \mathcal{H} , which is the intersection of all halfspaces. In \mathbb{R}^d , we need to sweep $O(n^d)$ directions (each determined by d points). So the polytope \mathcal{H} may have $O(n^d)$ faces. Using the dual approach, we can compute \mathcal{K} in $O(n^d)$ time (linear in the number of points in the dual space) as well. Overall, the running time is $O(n^d)$.

3.2.3 A Nearly Linear Time Algorithm for Constructing (ε, τ) -quant-kernels

We describe a nearly linear time algorithm for constructing an (ε, τ) -QUANT-KERNEL in the existential uncertainty model. As mentioned before, the algorithm in Section 3.2.1 takes linear time. So we only need a nearly linear time algorithm for constructing \mathcal{H} (and \mathcal{K}). Note that \mathcal{H} is the set of points in \mathbb{R}^d with Tukey depth at least $\ln(2/\tau)$. One tempting idea is to utilize the notion of ε -approximation (which can be obtained by sampling) to compute the approximate Tukey depth for the points, as done in [54]. However, a careful examination of this approach shows that the sample size needs to be as large as $O(\lambda(\mathcal{P}))$ (to ensure that for every halfspace, the difference between the real weight and the sample weight is less than, say $0.1 \ln(2/\tau)$). Another useful observation is that only points with small (around $\ln(2/\tau)$) Turkey depth are relevant in constructing \mathcal{H} . Hence, we can first sample an ε -approximation of very small size (say $k = O(\log n)$), and use it to quickly identify the region \mathcal{H}_1 in which all points have large (i.e., $\lambda(\mathcal{P})/k$) Turkey depth (so $\mathcal{H}_1 \subseteq \mathcal{H}$). Then, we can delete all points inside \mathcal{H}_1 and focus on the remaining points. Ideally, the total weight of the remaining points can be reduced significantly and a random sample of the same size k would give an ε' -approximation of the remaining points for some $\varepsilon' < \varepsilon$. We repeat the above until the total weight of the remaining points reduces to a constant, and then a constant size sample suffices. However, it is possible that all points have fairly small Tukey depth (consider the case where all points are in convex position), and no point can be removed. To resolve the issue, we use the idea in Lemma 31: there is a convex set \mathcal{K}_1 slightly larger than \mathcal{H}_1 such that the weight of points outside \mathcal{K}_1 is much smaller. Hence, we can make progress by deleting all points inside \mathcal{K}_1 . Since \mathcal{K}_1 is only slightly larger than \mathcal{H}_1 , we do not lose too much in terms of the distance. Our algorithm carefully implements the above iterative sampling idea.

For ease of exposition, we first focus on \mathbb{R}^2 . Consider the Poissonized instance of \mathcal{P} . We would like to find two convex sets \mathcal{H} and \mathcal{K} satisfying the following properties.

- P1. Assume without loss of generality that the origin is in \mathcal{H} . We require that $\frac{1}{1+\varepsilon}\mathcal{K} \subseteq \mathcal{H} \subseteq \mathcal{K}$.
- P2. For a direction $u \in \mathbb{S}^1$, we use $H(\mathcal{H}, u)$ to denote the halfplane which does not contain \mathcal{H} and whose boundary is the supporting line of \mathcal{H} with normal direction u . We require that $\lambda(H(\mathcal{H}, u)) = \sum_{v \in H(\mathcal{H}, u)} \lambda_v \geq \ln(2/\tau)$ for all directions $u \in \mathbb{S}^1$.
- P3. $\lambda(\overline{\mathcal{K}}) = \tilde{O}(1/\sqrt{\varepsilon})$.

By a careful examination of our analysis in Section 3.2.2, we can see the above properties are all we need for the analysis.

Let \mathcal{H}^* denote the \mathcal{H} found using the exact algorithm in Section 3.2.2. We use the following set of parameters:

$$z = O(\log n), \quad \varepsilon_1 = O\left(\frac{\varepsilon}{\log n}\right), \quad \varepsilon_2 = O\left(\sqrt{\frac{\varepsilon}{\log n}}\right).$$

Our algorithm proceeds in rounds. Initially, let $\mathcal{H}_0 = \text{ConvH}(\{v \in \mathcal{P} \mid \lambda_v \geq \ln(2/\tau)\})$. In round i (for $1 \leq i \leq z$), we construct two convex sets \mathcal{H}_i and \mathcal{K}_i such that

- 1. $\mathcal{H}_0 \subseteq \mathcal{K}_0 \subseteq \mathcal{H}_1 \subseteq \mathcal{K}_1 \subseteq \dots \subseteq \mathcal{H}_z \subseteq \mathcal{K}_z$;
- 2. $\frac{1}{1+\varepsilon_1}\mathcal{K}_i \subseteq \mathcal{H}_i \subseteq \mathcal{K}_i$ (\mathcal{K}_i and \mathcal{H}_i are very close to each other);
- 3. $\frac{1}{(1+\varepsilon_1)^i}\mathcal{H}_i \subseteq \mathcal{H}^*$ (\mathcal{H}_i is almost contained in \mathcal{H}^*);
- 4. $\lambda(\mathcal{P} \cap \overline{\mathcal{K}}_i) \leq \frac{1}{2}\lambda(\mathcal{P} \cap \overline{\mathcal{K}}_{i-1})$ (the total weight outside \mathcal{K}_i reduces by a factor of at least one half).

We repeat the above process until $\lambda(\overline{\mathcal{K}}_i) \leq \tilde{O}(1/\sqrt{\varepsilon})$.

Before spelling out the details of our algorithm, we need a few definitions.

Definition 34. For a set P of weighted points in \mathbb{R}^d , we use $\text{TK}(P, \gamma)$ to denote the set of points $x \in \mathbb{R}^d$ with Tukey depth at least γ . It is known that $\text{TK}(P, \gamma)$ is convex (see e.g., [54]). By this definition, $\mathcal{H}^* = \text{TK}(\mathcal{P}, \ln(2/\tau))$.

Recall the definition of ε -approximation from Theorem 23. By Theorem 23 (or Theorem 26), we can see that a set of $O(\varepsilon^{-2} \log(1/\delta))$ sampled points is an $\varepsilon\lambda(P)$ -approximation with probability $1 - \delta$.

We are ready to describe the details of our algorithm. Initially $\mathcal{H}_0 = \text{ConvH}(\{v \in \mathcal{P} \mid \lambda_v \geq \ln(2/\tau)\})$ (obviously $\mathcal{H}_0 \subseteq \mathcal{H}^*$). Compute a deterministic ε_1 -kernel $C_{\mathcal{H}_0}$ of \mathcal{H}_0 and let $\mathcal{K}_0 = (1 + \varepsilon_1)\text{ConvH}(C_{\mathcal{H}_0})$. Delete all point in $\mathcal{P} \cap \mathcal{K}_0$ and let \mathcal{P}_1 be the remaining points in \mathcal{P} (i.e., $\mathcal{P}_1 = \mathcal{P} \cap \overline{\mathcal{K}}_0$). Let $\text{Ver}(\mathcal{K}_0)$ denote all vertices of \mathcal{K}_0 (notice that some of them may not be original points in \mathcal{P}).

Now, suppose we describe the i th round for general $i > 1$. We have the remaining vertices in \mathcal{P}_{i-1} and $\text{Ver}(\mathcal{K}_{i-1})$. Let each point $v \in \mathcal{P}_i$ has the same old weight λ_v and each point in $\text{Ver}(\mathcal{K}_{i-1})$ has weight $+\infty$ (to make sure every point in \mathcal{K}_{i-1} has Turkey depth $+\infty$). Using random sampling on \mathcal{P}_i , obtain an $\varepsilon_2 \lambda(\mathcal{P}_i)$ -approximation \mathcal{E}_i (of size $L = O(\varepsilon_2^{-2})$ for \mathcal{P}_i). Then compute (using the brute-force algorithm described in Section 3.2.2)

$$\mathcal{H}_i = \text{TK}(\mathcal{E}_i \cup \text{Ver}(\mathcal{K}_{i-1}), \max\{4\varepsilon_2 \lambda(\mathcal{P}_i), 2 \ln(2/\tau)\}).$$

Note that $\mathcal{K}_{i-1} \subseteq \mathcal{H}_i$. Compute a deterministic ε_2 -kernel $C_{\mathcal{H}_i}$ of \mathcal{H}_i and let $\mathcal{K}_i = (1 + \varepsilon_1) \text{ConvH}(C_{\mathcal{H}_i})$ (hence $\text{ConvH}(C_{\mathcal{H}_i}) \subseteq \mathcal{H}_i \subseteq \mathcal{K}_i$). Then, we delete all points in $\mathcal{P} \cap \mathcal{K}_i$ and add all vertices of \mathcal{K}_i (denoted as $\text{Ver}(\mathcal{K}_i)$). Let \mathcal{P}_{i+1} be the remaining points in $\mathcal{P} \cap \bar{\mathcal{K}}_i$.

Our algorithm terminates when $\lambda(\mathcal{P}_i) \leq \tilde{O}(1/\sqrt{\varepsilon})$. Suppose the last round is z . Finally, we let $\mathcal{H} = \frac{1}{(1+\varepsilon_1)^z} \mathcal{H}_z$ and $\mathcal{K} = \mathcal{K}_z$.

First we show the algorithm terminates after at most a logarithmic number of rounds.

Lemma 35. $z = O(\log n)$.

Proof. If $\mathcal{H}_i = \text{TK}(\mathcal{E}_i \cup \text{Ver}(\mathcal{K}_{i-1}), 2 \ln(2/\tau))$, then we stop since $\lambda(\mathcal{P}_{i+1}) \leq \tilde{O}(1/\sqrt{\varepsilon})$ by Lemma 31. Thus, we only need to bound the number of iterations where $\mathcal{H}_i = \text{TK}(\mathcal{E}_i \cup \text{Ver}(\mathcal{K}_{i-1}), 4\varepsilon_2 \lambda(\mathcal{P}_i))$.

Initially, it is not hard to see that $\lambda(\mathcal{P}_1) \leq n \ln(2/\tau)$. Using Lemma 31, we can see that $\lambda(\mathcal{P}_{i+1}) = \lambda(\mathcal{P} \cap \bar{\mathcal{K}}_i) \leq O(5\varepsilon_2 \lambda(\mathcal{P}_i)/\sqrt{\varepsilon_1}) \leq \lambda(\mathcal{P}_i)/2$ for the constant defining ε_1 sufficiently large. Hence, $\lambda(\mathcal{P}_i) \leq \lambda(\mathcal{P})/2^i$. \square

We need to show \mathcal{H} and \mathcal{K} satisfy P1, P2 and P3. P3 is quite obvious by our algorithm. It is also not hard to see P1 since $(1 + \varepsilon_1)^{z+1} \leq 1 + \varepsilon$ and

$$\frac{1}{(1 + \varepsilon)} \mathcal{K}_z \subseteq \frac{1}{(1 + \varepsilon_1)^{z+1}} \mathcal{K}_z \subseteq \frac{1}{(1 + \varepsilon_1)^z} \mathcal{H}_z = \mathcal{H} \subseteq \mathcal{H}_z \subseteq \mathcal{K}_z = \mathcal{K}.$$

The most difficult part is to show P2 holds: For every direction $u \in \mathbb{S}^1$, $\lambda(H(\mathcal{H}, u)) = \sum_{v \in H(\mathcal{H}, u)} \lambda_v \geq \ln(2/\tau)$. In fact, we show that $\mathcal{H} \subseteq \mathcal{H}^*$, from which P2 follows trivially, which suffices to prove the following lemma.

Lemma 36. $\frac{1}{(1+\varepsilon_1)^i} \mathcal{H}_i \subseteq \mathcal{H}^*$ for all $0 \leq i \leq z$. In particular, $\mathcal{H} \subseteq \mathcal{H}^*$.

Proof. We prove the lemma by induction. $\mathcal{H}_0 \subseteq \mathcal{H}^*$ clearly satisfies the lemma. For ease of notation, we let $\eta = 1/(1 + \varepsilon_1)$. Suppose the lemma is true for \mathcal{H}_{i-1} , from which we can see that

$$\eta^i \mathcal{K}_{i-1} \subseteq \eta^{i-1} \mathcal{H}_{i-1} \subseteq \mathcal{H}^*.$$

Now we show the lemma holds for \mathcal{H}_i . Consider the i th round. Let \mathcal{E}_i be an $\varepsilon_2 \lambda(\mathcal{P}_i)$ -approximation for $\mathcal{P}_i = \mathcal{P} \cap \bar{\mathcal{K}}_{i-1}$ and $\mathcal{H}_i = \text{TK}(\mathcal{E}_i \cup \text{Ver}(\mathcal{K}_{i-1}), 4\varepsilon_2 \lambda(\mathcal{P}_i))$. Fix an arbitrary direction $u \in \mathbb{S}^1$ (w.l.o.g., assume that $u = (0, -1)$, i.e., the downward direction), let $H(\eta^i \mathcal{H}_i, u)$ be a halfplane whose boundary is tangent to $\eta^i \mathcal{H}_i$. It suffices to show that $\lambda(H(\eta^i \mathcal{H}_i, u)) = \sum_{v \in H(\eta^i \mathcal{H}_i, u)} \lambda_v \geq \ln(2/\tau)$. We move a sweep line ℓ_u orthogonal to u , along the direction u (i.e., from top to bottom), to sweep through the points in $\mathcal{P}_i \cap \text{Ver}(\mathcal{K}_{i-1})$ until the total weight we have swept is at least $\ln(2/\tau)$. We distinguish two cases:

1. ℓ_u hits a point v in $\text{Ver}(\mathcal{K}_{i-1})$ (recall the weight for such point is $+\infty$). We can see that v is the topmost point of \mathcal{K}_{i-1} and \mathcal{H}_i (or equivalently, ℓ_u is also a supporting line for \mathcal{H}_i). Since $\eta^i \mathcal{K}_{i-1} \subseteq \mathcal{H}^*$ by the induction hypothesis, the topmost point of $\eta^i \mathcal{K}_{i-1}$ is lower than that for \mathcal{H}^* . The topmost point of $\eta^i \mathcal{K}_{i-1}$ is also the highest point of $\eta^i \mathcal{H}_i$, from which we can see $H(\eta^i \mathcal{H}_i, u)$ is lower than $H(\mathcal{H}^*, u)$, which implies that $\lambda(H(\eta^i \mathcal{H}_i, u)) \geq \ln(2/\tau)$.
2. ℓ_u stops moving when it hits an original point in \mathcal{P}_i . Since $\max\{3\varepsilon_2 \lambda(\mathcal{P}_i), 2 \ln(2/\tau) - \varepsilon_2 \lambda(\mathcal{P}_i)\} > \ln(2/\tau)$, by definition of \mathcal{H}_i , $H(\mathcal{H}_i, u)$ can not be higher than ℓ_u . The boundary of $H(\eta^i \mathcal{H}_i, u)$ is even lower, from which we can see $\lambda(H(\eta^i \mathcal{H}_i, u)) \geq \ln(2/\tau)$.

Hence, every point in $\eta^i \mathcal{H}_i$ has Tukey depth at least $\ln(2/\tau)$, which implies the lemma. \square

Running time: In each round, we compute in linear time an ε_2 -approximation \mathcal{E}_i of size $O(\varepsilon_2^{-2} \log(1/\delta)) = \text{polylog}(n)$ (with $\delta = \text{poly}(n)$ to ensure each probabilistic event succeeds with high probability). \mathcal{K}_i is a dilation of an ε_1 -kernel. So the size of $\text{Ver}(\mathcal{K}_i)$ is at most $1/\sqrt{\varepsilon_1} = O(\log^{1/2} n)$. Deciding whether a point is inside \mathcal{K}_i can be solved in $\text{polylog}(n)$ time, by a linear program with $|\text{Ver}(\mathcal{K}_i)|$ variables. To compute \mathcal{H}_i , we can use the brute-force algorithm described in Section 3.2.2, which takes $\text{poly}(|\mathcal{E}_i \cap \text{Ver}(\mathcal{K}_{i-1})|) = \text{polylog}(n)$ time. There are logarithmic number of rounds. So the overall running time is $O(n \text{polylog} n)$.

Higher Dimension: Our algorithm can be easily extended to \mathbb{R}^d for any constant $d > 2$. In \mathbb{R}^d , we let $\varepsilon_1 = O(\varepsilon/\log n)$ and $\varepsilon_2 = O(\varepsilon/\log n)^{(d-1)/2}$. With the new parameters, we can easily check that Lemma 35 still holds. We can construct an (ε, τ) -QUANT-KERNEL of size $\min\{O(\tau^{-2} \max\{\lambda^2, \lambda^4\} \log(1/\delta)), O(\tau^{-2} \varepsilon^{-(d-1)} \log(1/\delta) \text{polylog}(1/\tau))\}$. The first term is from Theorem 30 and the second from the higher-dimensional extension to Theorem 32. Now, let us examine the running time. In \mathbb{R}^d , $|\text{Ver}(\mathcal{K}_i)|$ is at most $\varepsilon_1^{-(d-1)/2} = O(\log^{(d-1)/2} n)$. So deciding whether a point is inside \mathcal{K}_i can be solved in $\log^{O(d)}(n)$ time. Computing \mathcal{H}_i takes $\log^{O(d)}(n)$ time using the brute-force algorithm. So the overall running time is $O(n \log^{O(d)} n)$.

In summary, we obtain the following theorem for (ε, τ) -QUANT-KERNEL.

Theorem 6. (restated) \mathcal{P} is a set of uncertain points in \mathbb{R}^d with existential uncertainty. Let $\lambda = \sum_{v \in \mathcal{P}} (-\ln(1 - p_v))$. There exists an (ε, τ) -QUANT-KERNEL for \mathcal{P} , which consists of a set of independent uncertain points of cardinality $\min\{\tilde{O}(\tau^{-2} \max\{\lambda^2, \lambda^4\}), \tilde{O}(\varepsilon^{-(d-1)} \tau^{-2})\}$. The algorithm for constructing such a coresot runs in $\tilde{O}(n \log^{O(d)} n)$ time.

3.3 (ε, τ) -quant-kernel Under the Subset Constraint

We show it is possible to construct an (ε, τ) -QUANT-KERNEL in the existential model under the β -assumption: each possible location realizes a point with a probability at least β , where $\beta > 0$ is some fixed constant.

Theorem 37. Under the β -assumption, there is an (ε, τ) -QUANT-KERNEL in \mathbb{R}^d , which is of size $O(\mu^{-(d-1)/2} \log(1/\mu))$ and satisfies the subset constraint, in the existential uncertainty model, where $\mu = \min\{\varepsilon, \tau\}$.

In fact, the algorithm is exactly the same as constructing an ε -EXP-KERNEL and the proof of the above theorem is implicit in the proof of Theorem 20.

4 (ε, r) -fpow-kernel Under the β -Assumption

In this section, we show an (ε, r) -FPOW-KERNEL exists in the existential uncertainty model under the β -assumption. Recall that the function $T_r(P, u) = \max_{v \in P} \langle u, v \rangle^{1/r} - \min_{v \in P} \langle u, v \rangle^{1/r}$. For ease of notation, we write $\mathbb{E}[T_r(\mathcal{P}, u)]$ to denote $\mathbb{E}_{P \sim \mathcal{P}}[T_r(P, u)]$. Our goal is to find a set \mathcal{S} of stochastic points such that for all directions $u \in \mathcal{P}^*$, we have that $\mathbb{E}[T_r(\mathcal{S}, u)] \in (1 \pm \varepsilon)\mathbb{E}[T_r(\mathcal{P}, u)]$.

Our construction of \mathcal{S} is almost the same as that in Section 3.1. Suppose we sample N (fixed later) independent realizations and take the ε_0 -kernel for each of them. Suppose they are $\{\mathcal{E}_1, \dots, \mathcal{E}_N\}$ and we associate each a probability $1/N$. We denote the resulting (ε, r) -FPOW-KERNEL by \mathcal{S} . Hence, for any direction $u \in \mathcal{P}^*$, $\mathbb{E}[T_r(\mathcal{S}, u)] = \frac{1}{N} \sum_{i=1}^N T_r(\mathcal{E}_i, u)$ and we use this value as the estimation of $\mathbb{E}[T_r(\mathcal{P}, u)]$. Now, we show \mathcal{S} is indeed an (ε, r) -FPOW-KERNEL.

Recall that we use $\mathcal{E}(P)$ to denote the deterministic ε -kernel for any realization $P \sim \mathcal{P}$. We first compare \mathcal{P} with the random set $\mathcal{E}(P)$.

Lemma 38. *For any $t \geq 0$ and any direction $u \in \mathcal{P}^*$, we have that*

$$(1 - \varepsilon/2)\mathbb{E}[T_r(\mathcal{P}, u)] \leq \mathbb{E}_{P \sim \mathcal{P}}[T_r(\mathcal{E}(P), u, r)] \leq \mathbb{E}[T_r(\mathcal{P}, u)].$$

Proof. By Lemma 4.6 in [7], we have that $(1 - \varepsilon/2)T_r(P, u) \leq T_r(\mathcal{E}(P), u) \leq T_r(P, u)$. The lemma follows by combining all realizations. \square

Now we show that \mathcal{S} is an (ε, r) -FPOW-KERNEL of $\mathcal{E}(P)$. We first prove the following lemma. The proof is almost the same as that of Lemma 24, and can be found in Appendix C.

Lemma 39. *Let $N = O(\varepsilon_1^{-2} \varepsilon_0^{-(d-1)/2} \log(1/\varepsilon_0))$, where $\varepsilon_0 = (\varepsilon/4(r-1))^r$, $\varepsilon_1 = \varepsilon\beta^2$. For any $t \geq 0$ and any direction $u \in \mathcal{P}^*$, we have that*

$$\Pr_{P \sim \mathcal{S}}[\max_{v \in P} \langle u, v \rangle^{1/r} \geq t] \in \Pr_{P \sim \mathcal{P}}[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t] \pm \varepsilon_1/4, \text{ and}$$

$$\Pr_{P \sim \mathcal{S}}[\min_{v \in P} \langle u, v \rangle^{1/r} \geq t] \in \Pr_{P \sim \mathcal{P}}[\min_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t] \pm \varepsilon_1/4.$$

Lemma 40. *Let $N = O(\beta^{-4} \varepsilon^{-(rd-r+4)/2} \log(1/\varepsilon))$ and $\varepsilon_0 = (\varepsilon/4(r-1))^r$. \mathcal{S} constructed above is an (ε, r) -FPOW-KERNEL in \mathbb{R}^d .*

Proof. Fix a direction $u \in \mathcal{P}^*$. Let $A = \max_{v \in \mathcal{P}} \langle u, v \rangle^{1/r}$, $B = \min_{v \in \mathcal{P}} \langle u, v \rangle^{1/r}$. We observe that $B \leq \max_{v \in P} \langle u, v \rangle^{1/r} \leq A$ for any realization $P \sim \mathcal{P}$. We also need the following

basic fact about the expectation: For a random variable X , if $\Pr[X \geq a] = 1$, then $\mathbb{E}[X] = \int_b^\infty \Pr[X \geq x]dx + b$ for any $b \leq a$. Thus, we have that

$$\begin{aligned}\mathbb{E}_{P \sim \mathcal{P}}[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r}] &= \int_B^A \Pr_{P \sim \mathcal{P}}[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq x]dx + B \\ &\leq \int_B^A \Pr_{P \sim \mathcal{S}}[\max_{v \in P} \langle u, v \rangle^{1/r} \geq x]dx + B + \varepsilon_1(A - B)/4 \\ &= \mathbb{E}_{P \sim \mathcal{S}}[\max_{v \in P} \langle u, v \rangle^{1/r}] + \varepsilon_1(A - B)/4,\end{aligned}$$

where the first inequality is due to Lemma 39. Similarly, we can show the following two inequalities:

$$\begin{aligned}\mathbb{E}_{P \sim \mathcal{S}}[\max_{v \in P} \langle u, v \rangle^{1/r}] &\in \mathbb{E}_{P \sim \mathcal{P}}[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r}] \pm \varepsilon_1(A - B)/4, \\ \mathbb{E}_{P \sim \mathcal{S}}[\min_{v \in P} \langle u, v \rangle^{1/r}] &\in \mathbb{E}_{P \sim \mathcal{P}}[\min_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r}] \pm \varepsilon_1(A - B)/4.\end{aligned}$$

Recall that $T_r(P, u) = \max_{v \in P} \langle u, v \rangle^{1/r} - \min_{v \in P} \langle u, v \rangle^{1/r}$. By the linearity of expectation, we conclude that

$$\mathbb{E}[T_r(\mathcal{P}, u)] \in \mathbb{E}_{P \sim \mathcal{P}}[T_r(\mathcal{E}(P), u)] \pm \varepsilon_1(A - B)/2.$$

Combining Lemma 38, we have that $\mathbb{E}[T_r(\mathcal{S}, u)] \in (1 \pm \varepsilon/2)\mathbb{E}[T_r(\mathcal{P}, u)] \pm \varepsilon_1(A - B)/2$. By the β -assumption, we know that $\mathbb{E}[T_r(\mathcal{P}, u)] \geq \beta^2(A - B)$. Thus, $\varepsilon_1(A - B)/2 \leq \frac{\varepsilon}{2}\mathbb{E}[T_r(\mathcal{P}, u)]$, and $\mathbb{E}[T_r(\mathcal{S}, u)] \in (1 \pm \varepsilon)\mathbb{E}[T_r(\mathcal{P}, u)]$. \square

Running time: In each sample, the size of a deterministic ε_0 -kernel \mathcal{E}_i is at most $O(\varepsilon_0^{-(d-1)/2})$. Note that constructing an ε_0 -kernel can be solved in linear time. We take $O(\varepsilon_1^{-2}\varepsilon_0^{-(d-1)/2}\log(1/\varepsilon_0))$ samples in total. So the overall running time is $O(n\beta^{-4}\varepsilon^{-(rd-r+4)/2}\log(1/\varepsilon) + \text{poly}(1/\varepsilon)) = \tilde{O}(n\varepsilon^{-(rd-r+4)/2})$.

Note that each ε_0 -kernel contains $O(\varepsilon^{-r(d-1)/2})$ points. We take $N = O(\varepsilon_1^{-2}\varepsilon_0^{-(d-1)/2}\log(1/\varepsilon_0))$ independent samples. So the total size of (ε, r) -FPOW-KERNEL is $O(\beta^{-4}\varepsilon^{-(rd-r+2)}\log(1/\varepsilon))$. In summary, we obtain the following theorem.

Theorem 7. (restated) *An (ε, r) -FPOW-KERNEL of size $\tilde{O}(\varepsilon^{-(rd-r+2)})$ can be constructed in $\tilde{O}(n\varepsilon^{-(rd-r+4)/2})$ time in the existential uncertainty model under the β -assumption. In particular, the (ε, r) -FPOW-KERNEL consists of $N = \tilde{O}(\varepsilon^{-(rd-r+4)/2})$ point sets, each occurring with probability $1/N$ and containing $O(\varepsilon^{-r(d-1)/2})$ deterministic points.*

5 Applications

In this section, we show that our coresnet results for the directional width problem readily imply several coresnet results for other stochastic problems, just as in the deterministic setting. We introduce these stochastic problems and briefly summarize our results below.

5.1 Approximating the Extent of Uncertain Functions

We first consider the problem of approximating the extent of a set \mathcal{H} of uncertain functions. As before, we consider both the existential model and the locational model of uncertain functions.

1. In the existential model, each uncertain function h is a function in \mathbb{R}^d associated with a existential probability p_f , which indicates the probability that h presents in a random realization.
2. In the locational model, each uncertain function h is associated with a finite set $\{h_1, h_2, \dots\}$ of deterministic functions in \mathbb{R}^d . Each h_i is associated with a probability value $p(h_i)$, such that $\sum_i p(h_i) = 1$. In a random realization, h is independently realized to some h_i , with probability $p(h_i)$.

We use \mathcal{H} to denote the random instance, that is a random set of functions. We use $h \in \mathcal{H}$ to denote the event that the deterministic function h is present in the instance. For each point $x \in \mathbb{R}^d$, we let the random variable $\mathfrak{E}_{\mathcal{H}}(x) = \max_{h \in \mathcal{H}} h(x) - \min_{h \in \mathcal{H}} h(x)$ be the extent of \mathcal{H} at point x . Suppose \mathcal{S} is another set of uncertain functions. We say \mathcal{S} is the ε -EXP-KERNEL for \mathcal{H} if $(1 - \varepsilon)\mathfrak{E}_{\mathcal{H}}(x) \leq \mathfrak{E}_{\mathcal{S}}(x) \leq \mathfrak{E}_{\mathcal{H}}(x)$ for any $x \in \mathbb{R}^d$. We say \mathcal{S} is the (ε, τ) -QUANT-KERNEL for \mathcal{H} if $\Pr_{\mathcal{S} \sim \mathcal{S}}[\mathfrak{E}_{\mathcal{S}}(x) \leq t] \in \Pr_{\mathcal{H} \sim \mathcal{H}}[\mathfrak{E}_{\mathcal{H}}(x) \leq (1 \pm \varepsilon)t] \pm \phi$, for any $t \geq 0$ and any $x \in \mathbb{R}^d$.

Let us first focus on linear functions in \mathbb{R}^d . Using the *duality transformation* that maps linear function $y = a_1x_1 + \dots + a_dx_d + a_{d+1}$ to the point $(a_1, \dots, a_{d+1}) \in \mathbb{R}^{d+1}$, we can reduce the extent problem to the directional width problem in \mathbb{R}^{d+1} . Let \mathcal{H} be a set of uncertain linear functions (under either existential or locational model) in \mathbb{R}^d for constant d . From Theorem 14 and Corollary 11, we can construct a set \mathcal{S} of $O(n^{2d})$ deterministic linear functions in \mathbb{R}^d , such that $\mathfrak{E}_{\mathcal{S}}(x) = \mathbb{E}[\mathfrak{E}_{\mathcal{H}}(x)]$ for any $x \in \mathbb{R}^d$. Moreover, for any $\varepsilon > 0$, there exists an ε -EXP-KERNEL of size $O(\varepsilon^{-d/2})$ and an (ε, τ) -QUANT-KERNEL of size $\tilde{O}(\tau^{-2}\varepsilon^{-d})$. Using the standard linearization technique [7], we can obtain the following generalization for uncertain polynomials.

Theorem 41. *Let \mathcal{H} be a family of uncertain polynomials in \mathbb{R}^d (under either existential or locational model) that admits linearization of dimension k . We can construct a set \mathcal{M} of $O(n^{2k})$ deterministic polynomials, such that $\mathfrak{E}_{\mathcal{M}}(x) = \mathbb{E}[\mathfrak{E}_{\mathcal{H}}(x)]$ for any $x \in \mathbb{R}^d$. Moreover, for any $\varepsilon > 0$, there exists an ε -EXP-KERNEL of size $O(\varepsilon^{-k/2})$ and an (ε, τ) -QUANT-KERNEL of size $\min\{\tilde{O}(\tau^{-2} \max\{\lambda^2, \lambda^4\}), \tilde{O}(\varepsilon^{-k}\tau^{-2})\}$. Here $\lambda = \sum_{h \in \mathcal{H}} (-\ln(1 - p_h))$.*

Now, we consider functions of the form $u(x) = p(x)^{1/r}$ where $p(x)$ is a polynomial and r is a positive integer. We call such a function a *fractional polynomial*. We still use \mathcal{H} to denote the random set of fractional polynomials. Let $\mathcal{H}^* \subseteq \mathbb{R}^d$ be the set of points such that for any points $x \in \mathcal{H}^*$ and any function $u \in \mathcal{H}$, we have $u(x) \geq 0$. For each point $x \in \mathcal{H}^*$, we let the random variable $\mathfrak{E}_{r, \mathcal{H}}(x) = \max_{h \in \mathcal{H}} h(x)^{1/r} - \min_{h \in \mathcal{H}} h(x)^{1/r}$. We say another random set \mathcal{S} of functions is the (ε, r) -FPOW-KERNEL for \mathcal{H} if $(1 - \varepsilon)\mathfrak{E}_{r, \mathcal{H}}(x) \leq \mathfrak{E}_{r, \mathcal{S}}(x) \leq \mathfrak{E}_{r, \mathcal{H}}(x)$ for any $x \in \mathcal{H}^*$. By the duality transformation and Theorem 7, we can obtain the following result.

Theorem 42. *Let \mathcal{H} be a family of uncertain fractional polynomials in \mathbb{R}^d in the existential uncertainty model under the β -assumption. Further assume that each polynomial admits a linearization of dimension k . For any $\varepsilon > 0$, there exists an (ε, r) -FPOW-KERNEL of size $\tilde{O}(\varepsilon^{-(rk-r+2)})$. Furthermore, the (ε, r) -FPOW-KERNEL consists of $N = O(\varepsilon^{-(rk-r+4)/2})$ sets, each occurring with probability $1/N$ and containing $O(\varepsilon^{-r(k-1)/2})$ deterministic fractional polynomials.*

5.2 Stochastic Moving Points

We can extend our stochastic models to moving points. In the existential model, each point v is present with probability p_v and follows a trajectory $v(t)$ in \mathbb{R}^d when present ($v(t)$ is the position of v at time t). In the locational model, each point v is associated with a distribution of trajectories (the support size is finite) and the actual trajectory of v is a random sample for the distribution. Such uncertain trajectory models have been used in several applications in spatial databases [66]. For ease of exposition, we assume the existential model in the following. Suppose each trajectory is a polynomial of t with degree at most r . For each point v , any direction u and time t , define the polynomial $f_v(u, t) = \langle v(t), u \rangle$ and let \mathcal{H} include f_v with probability p_v . For a set \mathcal{P} of points, the directional width at time t is $\mathfrak{E}_{\mathcal{H}}(u, t) = \max_{v \in \mathcal{P}} f_v(u, t) - \min_{v \in \mathcal{P}} f_v(u, t)$. Each polynomial f_v admits a linearization of dimension $k = (r + 1)d - 1$. Using Theorem 41, we can see that there is a set M of $O(n^{2k})$ deterministic moving points, such that the directional width of M in any direction u is the same as the expected directional width of \mathcal{P} in direction u . Moreover, for any $\varepsilon > 0$, there exists an ε -EXP-KERNEL (which consists of only deterministic moving points) of size $O(\varepsilon^{-(k-1)/2})$ and an (ε, τ) -QUANT-KERNEL (which consists of both deterministic and stochastic moving points) of size $\tilde{O}(\varepsilon^{-k}\tau^{-2})$.

5.3 Shape Fitting Problems

Theorem 41 can be also applied to some stochastic variants of certain shape fitting problems. We first consider the following variant of the minimum enclosing ball problem over stochastic points. We are given a set \mathcal{P} of stochastic points (under either existential or locational model), find the center point c such that $\mathbb{E}[\max_{v \in \mathcal{P}} \|v - c\|^2]$ is minimized. It is not hard to see that the problem is equivalent to minimizing the expected area of the enclosing ball in \mathbb{R}^2 . For ease of exposition, we assume the existential model where v is present with probability p_v . For each point $v \in \mathcal{P}$, define the polynomial $h_v(x) = \|x\|^2 - 2\langle x, v \rangle + \|v\|^2$, which admits a linearization of dimension $d + 1$ [7]. Let \mathcal{H} be the family of uncertain polynomials $\{h_v\}_{v \in \mathcal{P}}$ (h_v exists with probability p_v). We can see that for any $x \in \mathbb{R}^d$, $\max_{v \in \mathcal{P}} \|x - v\|^2 = \max_{h_v \in \mathcal{H}} h_v(x)$. Using Theorem 41,¹⁰ we can see that there is a set M of $O(n^{2d+2})$ deterministic polynomials such that $\max_{h \in M} h(x) = \mathbb{E}[\max_{v \in \mathcal{P}} \|x - v\|^2]$ for any $x \in \mathbb{R}^d$ and a set S of $O(\varepsilon^{-(d+1)/2})$ deterministic polynomials such that $(1 - \varepsilon)\mathbb{E}[\max_{v \in \mathcal{P}} \|x - v\|^2] \leq \max_{h \in S} h(x) \leq \mathbb{E}[\max_{v \in \mathcal{P}} \|x - v\|^2]$ for

¹⁰ We can see from the proof that all results that hold for width/extent also hold for support function/maximum.

any $x \in \mathbb{R}^d$. We can store the set S instead of the original point set in order to answer the following queries: given a point v , return the expected length of the furthest point from v . The problem of finding the optimal center c can be also carried out over S , which can be done in $O(\varepsilon^{-O(d^2)})$ time: We can decompose the arrangement of n semialgebraic surfaces in \mathbb{R}^d into $O(n^{O(d+k)})$ cells of constant description complexity, where k is the linearization dimension (see e.g., [11]). By enumerating all those cells in the arrangement of S , we know which polynomials lie in the upper envelopes, and we can compute the minimum value in each such cell in constant time when d is constant.

The above argument can also be applied to the following variant of the spherical shell for stochastic points. We are given a set \mathcal{P} of stochastic points (under either existential or locational model). Our objective is to find the center point c such that $\mathbb{E}[\text{obj}(c)] = \mathbb{E}[\max_{v \in P} \|v - c\|^2 - \min_{v \in P} \|v - c\|^2]$ is minimized. The problem is equivalent to minimizing the expected area of the enclosing annulus in \mathbb{R}^2 . The objective can be represented as a polynomial of linearization dimension $k = d + 1$. Proceeding as for the enclosing balls, we can show there is a set S of $O(\varepsilon^{-(k-1)/2})$ deterministic polynomials such that $(1 - \varepsilon)\mathbb{E}[\text{obj}(c)] \leq \mathfrak{E}_S(x) \leq \mathbb{E}[\text{obj}(c)]$ for any $x \in \mathbb{R}^d$. We would like to make a few remarks here.

1. Let us take the minimum enclosing ball for example. If we examine the construction of set S , each polynomial $h \in S$ may *not* be of the form $h(x) = \|x\|^2 - 2\langle x, v \rangle + \|v\|^2$, therefore does not translate back to a minimum enclosing ball problem over deterministic points.
2. Another natural objective function for the minimum enclosing ball and the spherical shell problem would be the expected radius $\mathbb{E}[\max_{v \in P} d(v, c)]$ and the expected shell width $\mathbb{E}[\max_{v \in P} d(v, c) - \min_{v \in P} d(v, c)]$. However, due to the fractional powers (square roots) in the objectives, simply using an ε -EXP-KERNEL does not work. This is unlike the deterministic setting.¹¹ We leave the problem of finding small coresets for the spherical shell problem as an interesting open problem. However, under the β -assumption, we can use (ε, r) -FPOW-KERNELS to handle such fractional powers, as in the next subsection.

Theorem 8. (restated) *Suppose \mathcal{P} is a set of n independent stochastic points in \mathbb{R}^d under either existential or locational uncertainty model. There are linear time approximation schemes for the following problems: (1) finding a center point c to minimize $\mathbb{E}[\max_{v \in P} \|v - c\|^2]$; (2) finding a center point c to minimize $\mathbb{E}[\text{obj}(c)] = \mathbb{E}[\max_{v \in P} \|v - c\|^2 - \min_{v \in P} \|v - c\|^2]$. Note that when $d = 2$ the above two problems correspond to minimizing the expected areas of the enclosing ball and the enclosing annulus, respectively.*

5.4 Shape Fitting Problems (Under the β -assumption)

In this subsection, we consider several shape fitting problems in the existential model *under the β -assumption*. We show how to use Theorem 42 to obtain linear time approximation schemes for those problems.

¹¹ In particular, there is no stochastic analogue of Lemma 4.6 in [7].

1. (Minimum spherical shell) We first consider the minimum spherical shell problem. Given a set \mathcal{P} of stochastic points (under the β -assumption), our goal is to find the center point c such that $\mathbb{E}[\max_{v \in P} \|v - c\| - \min_{v \in P} \|v - c\|]$ is minimized. For each point $v \in P$, let $h_v(x) = \|x\|^2 - 2\langle x, v \rangle + \|v\|^2$, which admits a linearization of dimension $d + 1$. It is not hard to see that $\mathbb{E}[\max_{v \in P} \|v - c\|] = \mathbb{E}[\max_{v \in P} \sqrt{h_v(c)}]$ and $\mathbb{E}[\min_{v \in P} \|v - c\|] = \mathbb{E}[\min_{v \in P} \sqrt{h_v(c)}]$. Using Theorem 42, we can see that there are $N = \tilde{O}(\varepsilon^{-(d+3)})$ sets S_i , each containing $O(\varepsilon^{-(d+1)})$ fractional polynomial $\sqrt{h_v}$ s such that for all $x \in \mathbb{R}^d$,

$$\frac{1}{N} \sum_{i \in [N]} (\max_{S_i} \sqrt{h_v(x)} - \min_{S_i} \sqrt{h_v(x)}) \in (1 \pm \varepsilon) (\mathbb{E}[\max_{v \in P} \|v - x\|] - \mathbb{E}[\min_{v \in P} \|v - x\|]). \quad (6)$$

Note that our (ε, r) -FPOW-KERNEL satisfies the subset constraint. Hence, each function $\sqrt{h_v}$ corresponds to an original point in \mathcal{P} . So, we can store N point sets $P_i \subseteq \mathcal{P}$, with $|P_i| = O(\varepsilon^{-d})$ as the coreset for the original point set. By (6), an optimal solution for the coreset is an $(1 + \varepsilon)$ -approximation for the original problem.

Now, we briefly sketch how to compute the optimal solution for the coreset. Consider all points in $\cup_i P_i$. Consider the arrangement of $O(\varepsilon^{-O(d)})$ hyperplanes, each bisecting a pair of points in $\cup_i P_i$. For each cell C of the arrangement, for any point $v \in C$, the ordering of all points in $\cup_i P_i$ is fixed. We then enumerate all those cells in the arrangement and try to find the optimal center in each cell. Fix a cell C . For any point set P_i , we know which point is the furthest one and which point is the closest one from points in C_0 . Say they are $v_i = \arg \max_{v \in P_i} \|v - x\|$ and $v'_i = \arg \min_{v \in P_i} \|v - x\|$. Hence, our problem can be formulated as the following optimization problem:

$$\min_x \frac{1}{N} \sum_i (d_i - d'_i), \quad \text{s.t.} \quad d_i^2 = \|v_i - x\|^2, d_i'^2 = \|v'_i - x\|^2, d_i, d'_i \geq 0, \forall i \in [N]; x \in C_0.$$

The polynomial system has a constant number of variables and constraints, hence can be solved in constant time. More specifically, we can introduce a new variable t and let $t = \frac{1}{N} \sum_i (d_i - d'_i)$. All polynomial constraints define a semi-algebraic set. By using constructive version of Tarski-Seidenberg theorem, we can project out all variables except t and the resulting set is still a semi-algebraic set (which would be a finite collection of points and intervals in \mathbb{R}^1) (See e.g., [15]).

2. (Minimum enclosing cylinder, Minimum cylindrical shell) Let \mathcal{P} be a set of stochastic points in the existential uncertainty model under the β -assumption. Let $d(\ell, v)$ denote the distance between a point $v \in \mathbb{R}^d$ and a line $\ell \subset \mathbb{R}^d$. The goal for the minimum enclosing cylinder problem is to find a line ℓ such that $\mathbb{E}[\max_{v \in \mathcal{P}} d(\ell, v)]$ is minimized, while that for the minimum cylindrical shell problem is to minimize $\mathbb{E}[\max_{v \in \mathcal{P}} d(\ell, v) - \min_{v \in \mathcal{P}} d(\ell, v)]$. The algorithms for both problems are almost the same and we only sketch the one for the minimum enclosing cylinder problem.

We follow the approach in [7]. We represent a line $\ell \in \mathbb{R}^d$ by a $(2d - 1)$ -tuple $(x_1, \dots, x_{2d-1}) \in \mathbb{R}^{2d-1}$: $\ell = \{p + tq \mid t \in \mathbb{R}\}$, where $p = (x_1, \dots, x_{d-1}, 0)$ is the intersection point of ℓ with the hyperplane $x_d = 0$ and $q = (x_d, \dots, x_{2d-1})$, $\|q\|^2 = 1$ is the orientation of ℓ . Then for any point $v \in \mathbb{R}^d$, we have that

$$d(\ell, v) = \|(p - v) - \langle p - v, q \rangle q\|,$$

where the polynomial $d^2(\ell, v)$ admits a linearization of dimension $O(d^2)$. Now, proceeding as for the minimum enclosing ball problem and using Theorem 42, we can obtain a coreset \mathcal{S} consisting $N = O(\varepsilon^{-O(d^2)})$ deterministic point sets $P_i \subseteq \mathcal{P}$.

We briefly sketch how to obtain the optimal solution for the coreset. We can also decompose \mathbb{R}^{2d-1} (a point x in the space with $\|(x_d, \dots, x_{2d-1})\| = 1$ represents a line in \mathbb{R}^d) into $O(\varepsilon^{-O(d^2)})$ semi-algebraic cells such that for each cell, the ordering of the points in \mathcal{S} (by their distances to a line in the cell) is fixed. Note that such a cell is a semi-algebraic cell. For a cell C , assume that $v_i = \arg \max_{v \in P_i} d(\ell, v_i)$ for all $i \in [N]$, where ℓ is an arbitrary line in C . We can formulate the problem as the following polynomial system:

$$\min_l \frac{1}{N} \sum_i d_i, \quad \text{s.t.} \quad d_i^2 = d^2(\ell, v_i), d_i \geq 0, \forall i \in [N]; \ell = (p, q) \in C_0, \|q\|^2 = 1.$$

Again the polynomial system has a constant number of variables and constraints. Thus, we can compute the optimum in constant time.

Theorem 9. (restated) Suppose \mathcal{P} is a set of n independent stochastic points in \mathbb{R}^d , each appearing with probability at least β , for some fixed constant $\beta > 0$. There are linear time approximation schemes for minimizing the expected radius (or width) for the minimum spherical shell, minimum enclosing cylinder, minimum cylindrical shell problems over \mathcal{P} .

6 Concluding Remarks

We initiate the study of constructing coresets for various stochastic geometric extent problems. Our work opens up several avenues for further research. One obvious further direction is to construct constant sized (ε, r) -FPOW-KERNELS efficiently without the β -assumption. A promising approach is the importance sampling, based on the sensitivity of functions, developed in [48]. Note that such a construction would lead to efficient approximation schemes for several shape fitting problems without the β -assumption. While the size bounds we obtained for ε -EXP-KERNELS are tight (match the deterministic setting), the bounds for (ε, τ) -QUANT-KERNELS may not. Thus obtaining better bounds for (ε, τ) -QUANT-KERNEL is an interesting open problem. Another interesting and important further direction is to extend the concept of coresets to other problems (e.g., clustering) over stochastic datasets.

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A Missing Details in Section 2

A.1 Details for Section 2.1

Lemma 17. We find an affine transform T in $O(2^{O(d)}n \log n)$ time, such that the convex polytope $M' = T(M)$ is α -fat for some constant α .

Proof. By the results in [14], we only need to construct an approximate bounding box, which can be done as follows: We first identify two points y_1 and y_2 in M such that their distance is a constant approximation of the diameter of M . Then we project the points in M to a hyperplane $H \in \mathbb{R}^{d-1}$ perpendicular to the line through y_1 and y_2 , and recursively

identify two points among the projected points as the approximate diameter. Hence, it suffices to show how to identify such two points y_1 and y_2 . Let $\delta = \arccos(1/2)$. Suppose we are working on \mathbb{R}^d . We compute a set \mathcal{I} of $O(\delta^{-(d-1)})$ points on the unit sphere \mathbb{S}^{d-1} such that for any point $v \in \mathbb{S}^{d-1}$, there is a point $u \in \mathcal{I}$ such that $\angle(u, v) \leq \delta$ (see e.g., [10, 16]). From Lemma 16, we know that we can compute for each direction $u \in \mathcal{I}$, the point $x(u) \in M$ that maximizes $\langle u, x(u) \rangle$ in $O(n \log n)$ time. For each $u \in \mathcal{I}$, compute both $x(u)$ and $x(-u)$, and pick the pair that maximizes $\|x(u) - x(-u)\|$. Now, we argue this is a constant approximation of the diameter. Suppose the diameter of M is (y_1, y_2) where $y_1, y_2 \in M$. Consider the direction $v = (y_1 - y_2)/\|y_1 - y_2\|$. Without loss of generality, assume $y_1 = \arg \max_y \langle y, v \rangle$ and $y_2 = \arg \max_y \langle y, -v \rangle$. Moreover, there is a direction $u \in \mathcal{I}$ such that $\angle(u, v) \leq \delta$. Therefore, we can get that

$$\begin{aligned} \omega(M, u) &= f(M, u) + f(M, -u) \geq \langle y_1, u \rangle + \langle y_2, -u \rangle \\ &= \langle u, y_1 - y_2 \rangle = \|y_1 - y_2\| \cos \angle(u, v) \geq \|y_1 - y_2\|/2. \end{aligned}$$

In the third equation, we use the simple fact that $\cos \angle(u, v) = \langle u, v \rangle / \|u\| \|v\|$. \square

Lemma 18. $S = \{x(u)\}_{u \in \mathcal{I}}$ is an ε -kernel for M' .

Proof. Consider an arbitrary direction $v \in \mathbb{S}^{d-1}$ with $\|v\| = 1$. Suppose the point $a \in M'$ maximizes $\langle v, a \rangle$ $b \in M'$ maximizes $\langle -v, b \rangle$. Hence, $\omega(M', v) = \langle v, a \rangle - \langle v, b \rangle = \langle v, a - b \rangle$. By the construction of \mathcal{I} , there is a direction $u \in \mathcal{I}$ (with $\|u\| = 1$) such that $\|u - v\| \leq \delta$. Then, we can see that

$$\begin{aligned} \omega(S, v) &\geq \langle v, x(u) \rangle - \langle v, x(-u) \rangle = \langle v, x(u) - x(-u) \rangle \\ &= \langle u, x(u) - x(-u) \rangle + \langle v - u, x(u) - x(-u) \rangle \\ &\geq \langle u, a - b \rangle - \|v - u\| \|x(u) - x(-u)\| \\ &= \langle v, a - b \rangle + \langle u - v, a - b \rangle - \|v - u\| \|x(u) - x(-u)\| \\ &\geq \langle v, a - b \rangle - \|v - u\| \|x(u) - x(-u)\| - \|u - v\| \|a - b\| \\ &\geq \omega(M', v) - O(\delta d) \geq (1 - \varepsilon) \omega(M', v) \end{aligned}$$

In the last and 2nd to last inequalities, we use the fact that M' is α -fat (i.e., $\alpha \bar{\mathcal{C}} \subset M' \subset \bar{\mathcal{C}}$). \square

A.2 Details for Section 2.2

Theorem 20. Under the β -assumption, there is an ε -EXP-KERNEL in \mathbb{R}^d (for $d = O(1)$), which is of size $O(\beta^{-(d-1)} \varepsilon^{-(d-1)/2} \log(1/\varepsilon))$ and satisfies the subset constraint, in the existential uncertainty model.

Proof. Our algorithm is inspired by the peeling idea in [9]. Let $\varepsilon_1 = \varepsilon \alpha \beta^2 / 4 \sqrt{d}$, where α is a constant defined later. We repeat the following for $L = O(\log_{1-\beta} \varepsilon_1) = O(\log(1/\varepsilon))$ rounds: In round i , we first compute an $(\varepsilon_1 / \sqrt{d})$ -kernel \mathcal{S}_i (of size $O((\sqrt{d}/\varepsilon_1)^{(d-1)/2}) =$

$O(\beta^{-(d-1)}\varepsilon^{-(d-1)/2})$ for the remaining points (in the deterministic sense) and then delete all points of \mathcal{S}_i . Let $\mathcal{S} = \cup_i \mathcal{S}_i$. Now, we show that \mathcal{S} is an ε -EXP-KERNEL for \mathcal{P} .

We first establish a lower bound of $\omega(\mathcal{P}, u)$ for any unit vector $u \in \mathbb{S}^{d-1}$. Assume without loss of generality that $\alpha\bar{\mathbb{C}} \subset \text{ConvH}(\mathcal{P}) \subset \bar{\mathbb{C}}$ where $\bar{\mathbb{C}} = [-1, 1]^d$ and α is a constant only depending on d . Since $\alpha\bar{\mathbb{C}} \subset \text{ConvH}(\mathcal{P})$, we know there is a point $v \in \text{ConvH}(\mathcal{P})$ such that $\langle u, v \rangle \geq \alpha$ and a different point $w \in \text{ConvH}(\mathcal{P})$ such that $\langle u, w \rangle \leq -\alpha$. Hence, we have that

$$\omega(\mathcal{P}, u) \geq \beta^2(\langle u, v \rangle - \langle u, w \rangle) \geq 2\alpha\beta^2.$$

Fix an arbitrary direction $u \in \mathbb{S}^{d-1}$. Now, we bound the difference between $f(\mathcal{P}, u)$ and $f(\mathcal{S}, u)$. We show that for any real value $x \in [-\sqrt{d}, \sqrt{d}]$,

$$\Pr_{P \sim \mathcal{P}}[f(P, u) \geq x] \leq \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x - \varepsilon_1] + \varepsilon_1. \quad (7)$$

In fact, a proof of the above statement provides a proof for Theorem 37 (i.e., \mathcal{S} is an (ε, τ) -QUANT-KERNEL as well).

Let $\mathcal{L}_{\mathcal{P}} = \{v_1, v_2, \dots, v_L\}$ be the set of L points $v \in \mathcal{P}$ that maximize $\langle v, u \rangle$ (i.e., the first L vertices in the canonical order w.r.t. u). Similarly, let $\mathcal{L}_{\mathcal{S}} = \{w_1, w_2, \dots, w_L\}$ be the set of L points $v \in \mathcal{S}$ that maximize $\langle v, u \rangle$. We distinguish two cases:

1. $\mathcal{L}_{\mathcal{P}} = \mathcal{L}_{\mathcal{S}}$: If $x \geq \langle u, v_L \rangle$, we can see that $\Pr_{P \sim \mathcal{P}}[f(P, u) \geq x] = \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x]$. If $x < \langle u, v_L \rangle$, both $\Pr_{P \sim \mathcal{P}}[f(P, u) \geq x]$ and $\Pr_{S \sim \mathcal{S}}[f(S, u) \geq x]$ are at least $1 - \prod_{v \in \mathcal{L}_{\mathcal{P}}} (1 - p_v) \geq 1 - (1 - \beta)^L \geq 1 - \varepsilon_1$.
2. Suppose j is the smallest index such that $v_j \neq w_j$. For $x > \langle u, v_j \rangle$, we can see that $\Pr_{P \sim \mathcal{P}}[f(P, u) \geq x] = \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x]$. Now, we focus on the case where $x \leq \langle u, v_j \rangle$. From the construction of \mathcal{S} , we can see that $\langle w_{j'}, u \rangle \geq \langle v_j, u \rangle - \varepsilon_1$ for all $j' \geq j$.¹² Hence, for $x \leq \langle u, v_j \rangle$, we can see that

$$\Pr_{S \sim \mathcal{S}}[f(S, u) \geq x - \varepsilon_1] \geq 1 - \prod_{v \in \mathcal{L}_{\mathcal{S}}} (1 - p_v) \geq 1 - \varepsilon_1.$$

So, in either case, (7) is satisfied. We also need the following basic fact about the expectation: For a random variable X , if $\Pr[X \geq a] = 1$, then $\mathbb{E}[X] = \int_a^\infty \Pr[X \geq x]dx + b$ for any $b \leq a$. Since $-\sqrt{d} \leq f(P, u) \leq \sqrt{d}$ for any realization P , we have

$$\begin{aligned} f(\mathcal{P}, u) &= \int_{-\sqrt{d}}^\infty \Pr_{P \sim \mathcal{P}}[f(P, u) \geq x]dx - \sqrt{d} \\ &\leq \int_{-\sqrt{d}}^\infty \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x - \varepsilon_1]dx + 2\sqrt{d}\varepsilon_1 - \sqrt{d} \\ &\leq \int_{-\sqrt{d}-\varepsilon_1}^\infty \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x]dx - \sqrt{d} - \varepsilon_1 + 3\sqrt{d}\varepsilon_1 \\ &= f(\mathcal{S}, u) + 3\sqrt{d}\varepsilon_1, \end{aligned}$$

¹² To see this, consider the round in which $w_{j'}$ is chosen. Let t be the vertex minimizing $\langle t, u \rangle$. As v_j is not chosen, we must have $\langle w_{j'}, u \rangle - \langle t, u \rangle \geq (1 - \varepsilon_1/\sqrt{d})(\langle v_j, u \rangle - \langle t, u \rangle)$.

where the only inequality is due to (7) and the fact that $\Pr_{P \sim \mathcal{P}}[f(P, u) \geq x] = \Pr_{S \sim \mathcal{S}}[f(S, u) \geq x] = 0$ for $x > 1$. Similarly, we can get that $f(\mathcal{S}, -u) \geq f(\mathcal{P}, -u) - 3\varepsilon_1\sqrt{d}$. By the choice of ε_1 , we have that $6\sqrt{d}\varepsilon_1 \leq \varepsilon \cdot 2\alpha\beta^2 \leq \varepsilon\omega(\mathcal{P}, u)$. Hence, $\omega(\mathcal{S}, u) \geq \omega(\mathcal{P}, u) - 6\sqrt{d}\varepsilon_1 \geq (1 - \varepsilon)\omega(\mathcal{P}, u)$. \square

A.3 Locational uncertainty

Similar results are possible for uncertain points with locational uncertainty. Now there are m possible locations, and thus $\binom{m}{2}$ hyperplanes Γ that partition \mathbb{R}^d . We can replicate all bounds in this setting, except that m replaces n in each bound. The main difficulty is in replicating Lemma 43 that given a direction u calculates the vertex of M ; for locational uncertain points this is described in Lemma 45. Moreover, the $O(m^2 \log m)$ bound for \mathbb{R}^2 is also carefully described in Lemma 46.

In the locational uncertainty model, Lemma 12 also holds with a stronger general position assumption. With the new general position assumption, it is straightforward to show that the gradient vector is different for two adjacent cones in $\mathbb{A}(\Gamma)$. Other parts of the proof is essentially the same as Lemma 12. The details can be found below. Theorem 4 also holds for the locational model without any change in the proof (the running time becomes $O(m \log^2 m)$).

Now, we prove that Lemma 12 also holds for the locational model. For this purpose, we need a stronger general position assumption: (1) For any $v \in \mathcal{P}$, $\sum_{s \in \mathcal{S}} p_{vs} \in (0, 1)$. This suggests that we need to consider the model with both existential and locational uncertainty. We can make this assumption hold by subtracting an infinitesimal value from each probability value without affecting the directional width in any essential way. (2) For any two points $v_1, v_2 \in \mathcal{P}$, two positions s_1, s_2 and two subsets of positions S_1, S_2 , $p_{v_1, s_1}(\sum_{s \in S_2} p_{v_2, s})s_1 \neq p_{v_2, s_2}(\sum_{s \in S_1} p_{v_1, s})s_2$ (this is indeed a general position assumption since we only have a finite number of equations to exclude but uncountable number of choices of the positions).

Lemma 12. (for the locational model). Assuming the locational model and the above general position assumption, the complexity of M is the same as the cardinality of $\mathbb{A}(\Gamma)$, i.e., $|M| = |\mathbb{A}(\mathcal{P})|$. Moreover, each cone $C \in \mathbb{A}(\mathcal{P})$ corresponds to exactly one vertex v of $\text{ConvH}(M)$ in the following sense: $\nabla f(M, u) = v$ for all $u \in \text{int } C$.

Proof. The proof is almost the same as that for Lemma 12 except that we need to show $f(M, u)$ is different for two adjacent cones in $\mathbb{A}(\Gamma)$. Again, let C_1, C_2 be two adjacent cones separated by some hyperplane in Γ . Suppose $u_1 \in \text{int } C_1$ and $u_2 \in \text{int } C_2$. Consider the canonical orders O_1 and O_2 of \mathcal{P} with respect to u_1 and u_2 respectively. W.l.o.g., assume that $O_1 = \{s_1, \dots, s_i, s_{i+1}, \dots, s_n\}$ and $O_2 = \{s_1, \dots, s_{i+1}, s_i, \dots, s_n\}$.

Using the notations from Lemma 45, $f(\mathcal{P}, u)$ can be computed by $\sum_{s \in \mathcal{S}, v \in \mathcal{P}} \Pr^R(v, s, u) \langle s, u \rangle$. Hence, $\nabla f(\mathcal{P}, u) = \sum_{s \in \mathcal{S}, v \in \mathcal{P}} \Pr^R(v, s, u) s$.

Suppose s_i is a possible location for v_1 and s_{i+1} is a possible location for v_2 . If $v_1 \neq v_2$, we have $\nabla f(\mathcal{P}, u_1) - \nabla f(\mathcal{P}, u_2) = \Pr_\emptyset^R(s_i, u_1) \cdot \left(s_i p_{v_1 s_i} \sum_{s' \in SR(s_i, u_1)} p_{v_2 s'} - s_{i+1} p_{v_2 s_{i+1}} \sum_{s' \in SR(s_i, u_2)} p_{v_1 s'} \right) \neq 0$.

0. If $v_1 = v_2 = v$, we have $\nabla f(\mathcal{P}, u_1) - \nabla f(\mathcal{P}, u_2) = \Pr_\emptyset^R(s_i, u_1) \cdot (s_i p_{vs_i} - s_{i+1} p_{vs_{i+1}}) \neq 0$. \square

B Missing Details in Section 3

Theorem 30. Let $\tau_1 = O(\frac{\tau}{\max\{\lambda, \lambda^2\}})$ and $N = O(\frac{1}{\tau_1^2} \log \frac{1}{\delta}) = O(\frac{\max\{\lambda^2, \lambda^4\}}{\tau^2} \log \frac{1}{\delta})$. With probability at least $1 - \delta$, for any $t \geq 0$ and any direction u , we have that $\Pr[\omega(\mathcal{S}, u) \leq t] \in \Pr[\omega(\mathcal{P}, u) \leq t] \pm \tau$.

Proof. Fix an arbitrary direction u (w.l.o.g., say it is the x-axis) and rename all points in \mathcal{P} as v_1, v_2, \dots, v_n as before. Consider the Poissonized instance of \mathcal{P} . Let v'_1, \dots, v'_N be the N points in \mathcal{S} (also sorted in nondecreasing order of their x-coordinates). Now, we create a coupling between all mass in \mathfrak{A} and that in \mathfrak{B} , as follows. We process all points in \mathfrak{A} from left to right, starting with v_1 . The process has N rounds. In each round, we assign exactly $1/N$ units of mass in \mathfrak{A} to a point in \mathfrak{B} . In the first round, if v_1 contains less than $1/N$ units of mass, we proceed to v_2, v_3, \dots, v_i until we reach $1/N$ units collectively. We split the last node v_i into two nodes v_{i1} and v_{i2} so that the mass contained in $v_1, \dots, v_{i-1}, v_{i1}$ is exactly $1/N$, and we assign those nodes to v'_1 . We start the next round with v_{i2} . If v_1 contains more than $1/N$ units of mass, we split v_1 into v_{11} (v_{11} contains $1/N$ units) and v_{12} and we start the second round with v_{12} . We repeat this process until all mass in \mathfrak{A} is assigned.

The above coupling can be viewed as a mass transportation from \mathfrak{A} to \mathfrak{B} . We will need one simple but useful property about this transportation: for any vertical line $x = t$, at most τ_1 units of mass are transported across the vertical line (by Theorem 26).

In the construction of the coupling, many nodes in \mathfrak{A} may be split. We rename them to be v_1, \dots, v_m (according to the order in which they are processed). The sequence v_1, \dots, v_m can be divided into N segments, each assigned to a point in \mathcal{S} . For a point v'_i in \mathcal{S} , let $\text{seg}(i)$ be the segment (the set of points) assigned to v'_i . For any node v and real $t > 0$, we use $H(v, t)$ to denote the right open halfplane defined by the vertical line $x = x(v) + t$, where $x(v)$ is the x-coordinate of v (see Figure 3).

Let X_i (Y_i resp.) be the Poisson distributed random variable corresponding to v_i (v'_i resp.) (i.e., $X_i \sim \text{Pois}(\lambda_{v_i})$ and $Y_i \sim \text{Pois}(\lambda/N)$) for all i . For any $H \subset \mathbb{R}^2$, we write $X(H) = \sum_{v_i \in H \cap \mathcal{P}} X_i$ and $Y(H) = \sum_{v'_i \in H \cap \mathcal{S}} Y_i$. We can rewrite $\Pr[\omega(\mathcal{S}, u) \leq t]$ as follows:

$$\begin{aligned} \Pr[\omega(\mathcal{S}, u) \leq t] &= \sum_{i=1}^N \Pr[v'_i \text{ is the leftmost point and } \omega(\mathcal{S}, u) \leq t] + \Pr[\text{no point in } \mathcal{S} \text{ appears}] \\ &= \sum_{i=1}^N \Pr[Y_i \neq 0] \Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right] \Pr[Y(H(v'_i, t)) = 0] + \Pr\left[\sum_{v'_i \in \mathcal{S}} Y_i = 0\right] \end{aligned} \quad (8)$$

Similarly, we can write that ¹³

$$\begin{aligned}
\Pr[\omega(\mathcal{P}, u) \leq t] &= \sum_{i=1}^m \Pr[X_i \neq 0] \Pr\left[\sum_{j=1}^{i-1} X_j = 0\right] \Pr[X(H(v_i, t)) = 0] + \Pr\left[\sum_{v_i \in \mathcal{P}} X_i = 0\right] \\
&= \sum_{i=1}^N \sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] \Pr\left[\sum_{j=1}^{k-1} X_j = 0\right] \Pr[X(H(v_k, t)) = 0] + \Pr\left[\sum_{v_i \in \mathcal{P}} X_i = 0\right]
\end{aligned} \tag{9}$$

We proceed by attempting to show each each summand of (9) is close to the corresponding one in (8). First, we can see that $\Pr[\sum_{v'_i \in \mathcal{S}} Y_i = 0] = \Pr[\sum_{v_i \in \mathcal{P}} X_i = 0]$ since both $\sum_{v'_i \in \mathcal{S}} Y_i$ and $\sum_{v_i \in \mathcal{P}} X_i$ follow the Poisson distribution $\text{Pois}(\lambda)$.

For any segment i , we can see that $\sum_{k \in \text{seg}(i)} \lambda_{v_k} = \lambda/N$. Moreover, we have $\lambda_{v_k} \leq \lambda/N \leq \tau/32$, thus $\exp(-\lambda_{v_k}) \in (1 - \lambda_{v_k}, (1 + \tau/16)(1 - \lambda_{v_k}))$.

$$\begin{aligned}
\sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] &= \sum_{k \in \text{seg}(i)} (1 - \exp(-\lambda_{v_k})) \in (1 \pm \frac{\tau}{16}) \sum_{k \in \text{seg}(i)} \lambda_{v_k} \\
&\subset (1 \pm \frac{\tau}{8})(1 - \exp(-\frac{\lambda}{N})) = (1 \pm \frac{\tau}{8})\Pr[Y_i \neq 0].
\end{aligned} \tag{10}$$

Then, we notice that for any $k \in \text{seg}(i)$ (i.e., v_k is in the segment assigned to v'_i), it holds that

$$\Pr\left[\sum_{j=1}^k X_j = 0\right] \in [e^{-i\lambda/N}, e^{-\lambda(i-1)/N}] \subset (1 \pm \frac{\tau}{8})e^{-\lambda(i-1)/N} = (1 \pm \frac{\tau}{8})\Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right]. \tag{11}$$

The first inequality holds because $\sum_{j=1}^k X_j \sim \text{Pois}(\sum_{j=1}^k \lambda_{v_j})$ and $\lambda(i-1)/N \leq \sum_{j=1}^k \lambda_{v_j} \leq \lambda i/N$.

If we can show that $\Pr[X(H(v_k, t)) = 0]$ is close to $\Pr[Y(H(v'_i, t)) = 0]$ for $k \in \text{seg}(i)$, we can finish the proof easily since each summand of (9) would be close to the corresponding one in (8). However, this is in general not true and we have to be more careful.

Recall that the sequence v_1, \dots, v_m is divided into N segments. Let $K = \lambda/\tau$. We say that the i th segment (say $\text{seg}(i) = \{v_j, v_{j+1}, \dots, v_k\}$) is a *good segment* if

$$E_i = \max\left\{\left|\mathfrak{B}(H(v'_i, t)) - \mathfrak{A}(H(v_j, t))\right|, \left|\mathfrak{B}(H(v'_i, t)) - \mathfrak{A}(H(v_k, t))\right|\right\} \leq \frac{1}{K}.$$

Otherwise, the segment is *bad*. For a good segment $\text{seg}(i)$ and any $k \in \text{seg}(i)$,

$$\begin{aligned}
\Pr[X(H(v_k, t)) = 0] &= \exp(-\lambda \mathfrak{A}(H(v_k, t))) \in \exp(-\lambda \mathfrak{B}(H(v'_i, t)) \pm \lambda/K) \\
&\subset \Pr[Y(H(v'_i, t)) = 0]e^{\pm \lambda/K} \subset \Pr[Y(H(v'_i, t)) = 0](1 \pm \tau/8).
\end{aligned} \tag{12}$$

¹³ Note that splitting nodes does not change the distribution of $\omega(\mathcal{P}, u)$: Suppose a node v (corresponding to r.v. X) was spit to two nodes v_1 and v_2 (corresponding to X_1 and X_2 resp.). We can see that $\Pr[X \neq 0] = \Pr[X_1 \neq 0 \text{ and } X_2 \neq 0] = \Pr[X_1 + X_2 \neq 0]$.

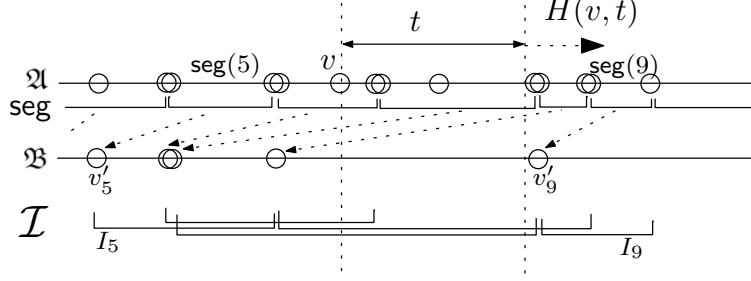


Figure 3: Illustration of the interval graph \mathcal{I} . For illustration purpose, co-located points (e.g., points that are split in \mathfrak{A}) are shown as overlapping points. The arrows indicate the assignment of the segments to the points in \mathfrak{B} . Theorem 26 ensures that any vertical line can not stab many intervals.

We use \mathbf{Gs} to denote the set of good segments and \mathbf{Bs} the set of bad segments. Now, we consider the summations in both (8) and (9) with only good segments. We have that

$$\begin{aligned}
& \sum_{i \in \mathbf{Gs}} \sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] \Pr\left[\sum_{j=1}^{k-1} X_j = 0\right] \Pr[X(H(v_k, t)) = 0] \\
& \in \sum_{i \in \mathbf{Gs}} \Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right] (1 \pm \tau/8) \Pr[Y(H(v'_i, t)) = 0] (1 \pm \tau/8) \Pr[Y_i \neq 0] (1 \pm \tau/8) \\
& \subset \sum_{i \in \mathbf{Gs}} \Pr[Y_i \neq 0] \Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right] \Pr[Y(H(v'_i, t)) = 0] \pm \tau/2,
\end{aligned}$$

where the first inequality is due to (11) and (12) and the second holds because (10)

Now, we show the total contributions of bad segments to both (8) and (9) are small. We partition all of the bad segments into $\log(1/\tau) + 1$ different sets $B_0, \dots, B_{\log \frac{1}{\tau} - 1}$. Let $B_i = \{i \mid \frac{2^i}{K} < E_i \leq \frac{2^{i+1}}{K}\}$ for $0 \leq i \leq \log(1/\tau) - 1$. Let $B_{\log \frac{1}{\tau}} = \{i \mid E_i > \frac{1}{\lambda}\}$.

With the above notations, we prove the following crucial inequality:

$$\sum_{i=0}^{\log \frac{1}{\tau}} |B_i| \cdot 2^i = O(\tau_1 N K). \quad (13)$$

Now, we prove (13). Consider all points v_1, \dots, v_m and v'_1, \dots, v'_n lying on the same x -axis. For each i (with $\text{seg}(i) = \{v_j, v_{j+1}, \dots, v_k\}$), we draw the minimal interval I_i that contains v'_i, v_j and v_k . If the i th segment is bad and belongs to B_j , we also say I_i is a *bad interval* of label j . All intervals $\{I_i\}_i$ define an interval graph \mathcal{I} . We can see that any vertical line can stab at most $\tau_1 N + 1$ intervals, because at most τ_1 unit of mass can be transported across the vertical line, and each interval is responsible for a transportation of exactly $1/N$ units of mass (except the one that intersects the vertical line). Hence, the interval graph \mathcal{I} can be colored with at most $\tau_1 N + 1$ colors (this is because the clique number of \mathcal{I} is at most

$\tau_1 N + 1$ and the chromatic number of an interval graph is the same as its clique number). Consider a color class C (which consists of a set of non-overlapping intervals). Imagine we move an interval I of length t along the x -axis from left to right. When the left endpoint of I passes through an bad interval of label j in C , by the definition of bad segments, the right endpoint of I passes through $\Omega(2^j N/K)$ segments. Suppose the color class C contains b_j bad segments in B_j . Since the right endpoint of I can pass through at most N segments, we have the following inequalities by summing over all labels:

$$\sum_{j=0}^{\log \frac{1}{\tau} - 1} b_j \cdot 2^j N/K \leq N.$$

Summing up all color classes, we obtain (13).

For B_j ($0 \leq j \leq \log \frac{1}{\tau}$), we can bound the total contribution as follows. By the definition of B_j , we can see that

$$\begin{aligned} & \sum_{i \in B_j} \sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] \Pr\left[\sum_{j=1}^{k-1} X_j = 0\right] \Pr[X(H(v_k, t)) = 0] \\ & \subset (1 \pm 5 \cdot 2^j \tau) \sum_{i \in B_j} \Pr[Y_i \neq 0] \Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right] \Pr[Y(H(v'_i, t)) = 0], \end{aligned}$$

Thus, the total contribution of bad segments in B_j ($0 \leq j \leq \log \frac{1}{\tau} - 1$) to the corresponding summands in ((8)-(9)) is at most

$$5 \cdot 2^j \tau \sum_{i \in B_j} \Pr[Y_i \neq 0] = 5|B_j| \cdot 2^j \tau \times (1 - \exp(-\frac{\lambda}{N})) = O(|B_j| 2^j \tau \lambda / N),$$

where $\Pr[Y_i \neq 0] = 1 - \exp(-\frac{\lambda}{N})$ (since $Y_i \sim \text{Pois}(\frac{\lambda}{N})$).

For $B_{\log \frac{1}{\tau}}$, the total contribution is bounded by the following.

$$\begin{aligned} & \left| \sum_{i \in B_{\log \frac{1}{\tau}}} \left(\sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] \Pr\left[\sum_{j=1}^{k-1} X_j = 0\right] \Pr[X(H(v_k, t)) = 0] \right. \right. \\ & \quad \left. \left. - \Pr[Y_i \neq 0] \Pr\left[\sum_{j=1}^{i-1} Y_j = 0\right] \Pr[Y(H(v'_i, t)) = 0] \right) \right| \\ & \leq \sum_{i \in B_{\log \frac{1}{\tau}}} \sum_{k \in \text{seg}(i)} \Pr[X_k \neq 0] + \sum_{i \in B_{\log \frac{1}{\tau}}} \Pr[Y_i \neq 0] \leq 3 \sum_{i \in B_{\log \frac{1}{\tau}}} \Pr[Y_i \neq 0] \\ & \leq 3|B_{\log \frac{1}{\tau}}| \times (1 - \exp(-\frac{\lambda}{N})) = O(|B_{\log \frac{1}{\tau}}| \lambda / N) \end{aligned}$$

Summing up all j and using (13), we obtain the following inequality .

$$\sum_{j=0}^{\log \frac{1}{\tau}} O(|B_j| 2^j \tau \lambda / N) = O(\tau_1 \tau \lambda K) \leq \frac{\tau}{4}.$$

This finishes the proof. \square

C Missing Details in Section 4

Lemma 39. Let $N = O(\varepsilon_1^{-2} \varepsilon_0^{-(d-1)/2} \log(1/\varepsilon_0))$, where $\varepsilon_0 = (\varepsilon/4(r-1))^r$, $\varepsilon_1 = \varepsilon \beta^2$. For any $t \geq 0$ and any direction $u \in \mathcal{P}^*$, we have that

$$\Pr_{P \sim \mathcal{S}} \left[\max_{v \in P} \langle u, v \rangle^{1/r} \geq t \right] \in \Pr_{P \sim \mathcal{P}} \left[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t \right] \pm \varepsilon_1/4, \text{ and}$$

$$\Pr_{P \sim \mathcal{S}} \left[\min_{v \in P} \langle u, v \rangle^{1/r} \geq t \right] \in \Pr_{P \sim \mathcal{P}} \left[\min_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t \right] \pm \varepsilon_1/4.$$

Proof. The argument is almost the same as that in Lemma 24. Let $L = O(\varepsilon_0^{-(d-1)/2})$. We still build a mapping g that maps each realization $\mathcal{E}(P)$ to a point in \mathbb{R}^{dL} , as follows: Consider a realization P of \mathcal{P} . Suppose $\mathcal{E}(P) = \{(x_1^1, \dots, x_d^1), \dots, (x_1^L, \dots, x_d^L)\}$ (if $|\mathcal{E}(P)| < L$, we pad it with $(0, \dots, 0)$). We let $g(\mathcal{E}(P)) = (x_1^1, \dots, x_d^1, \dots, x_1^L, \dots, x_d^L) \in \mathbb{R}^{dL}$. For any $t \geq 0$ and any direction $u \in \mathcal{P}^*$, note that $\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t$ holds if and only if there exists some $1 \leq i \leq |\mathcal{E}(P)|$ satisfies that $\sum_{j=1}^d x_j^i u_j \geq t^r$, which is equivalent to saying that point $g(\mathcal{E}(P))$ is in the union of the those $|\mathcal{E}(P)|$ half-spaces.

Let X be the image set of g . Let (X, \mathcal{R}^i) ($1 \leq i \leq L$) be a range space, where \mathcal{R}^i is the set of half spaces $\{\sum_{j=1}^d x_j^i u_j \geq t \mid u = (u_1, \dots, u_d) \in \mathbb{R}^d, t \geq 0\}$. Let $\mathcal{R}' = \{\cup r_i \mid r_i \in \mathcal{R}^i, i \in [L]\}$. Note that each (X, \mathcal{R}^i) has VC-dimension $d+1$. By Theorem 22, we have that the VC-dimension of (X, \mathcal{R}') is bounded by $O((d+1)L \lg L) = O(\varepsilon_0^{-(d-1)/2} \log(1/\varepsilon_0))$. Then by Theorem 23, for any t and any direction u , we have that $\Pr_{P \sim \mathcal{S}}[\max_{v \in P} \langle u, v \rangle^{1/r} \geq t] \in \Pr_{P \sim \mathcal{P}}[\max_{v \in \mathcal{E}(P)} \langle u, v \rangle^{1/r} \geq t] \pm \varepsilon_1/4$. The proof for the second statement is the same. \square

D Computing the Expected Direction Width

We handle both the existential and location model of uncertain points in this section. For any direction u , denote by $\omega(\mathcal{P}, u)$ the expected width of \mathcal{P} along the direction u , and $f(\mathcal{P}, u) = \mathbb{E}_{P \sim \mathcal{P}}[\max_{p \in P} \langle u, p \rangle]$ is the support function. Recall $\omega(\mathcal{P}, u) = f(\mathcal{P}, u) - f(\mathcal{P}, -u)$ by linearity of expectation.

D.1 Computing Expected Width for Existential Uncertainty

The existential model is a bit simpler and we handle that first. Recall in this model we let \mathcal{P} be a set of n uncertain points, and each point $v \in \mathcal{P}$ has a probability p_v . We have the following two lemmas.

Lemma 43. *For any direction u , we can compute $\omega(\mathcal{P}, u)$, $f(\mathcal{P}, u)$, and $\nabla f(\mathcal{P}, u)$ in $O(n \log n)$ time; if the points of \mathcal{P} are already sorted along the direction u , then we can compute them in $O(n)$ time.*

Proof. Consider any direction u . Without loss of generality, assume $\|u\| = 1$. In the following, we first show how to compute $f(\mathcal{P}, u)$. The value $f(\mathcal{P}, -u)$ can be computed in a similar manner and we ignore the discussion. After having $f(\mathcal{P}, u)$ and $f(\mathcal{P}, -u)$, $\omega(\mathcal{P}, u)$ can be computed immediately by $\omega(\mathcal{P}, u) = f(\mathcal{P}, u) - f(\mathcal{P}, -u)$. Finally, we will discuss how to compute $\nabla f(\mathcal{P}, u)$. Let $\rho(u)$ be the ray of direction u in the plane passing through the origin.

Consider a point $v \in \mathcal{P}$. Note that $\langle v, u \rangle$ is the coordinate of the perpendicular projection of v on $\rho(u)$. Denote by $\mathcal{P}^R(v, u)$ the subset of points $v' \in \mathcal{P}$ such that $v' >_u v$ (i.e., $\langle v', u \rangle > \langle v, u \rangle$). Denote by $\text{Pr}^R(v, u)$ the probability that v appears in a realization but all points of $\mathcal{P}^R(v, u)$ do not appear (i.e., $\langle v, u \rangle$ is the largest among all points of \mathcal{P} that appear in the realization). Hence, we have

$$\text{Pr}^R(v, u) = p_v \cdot \prod_{v' >_u v} (1 - p_{v'}). \quad (14)$$

Now $f(\mathcal{P}, u)$ can be seen as the expected largest coordinate of the projections of the points in \mathcal{P} on $\rho(u)$. According to the definition of $\text{Pr}^R(v, u)$, we have $f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u) \langle v, u \rangle$.

Based on the above discussion, we can compute $f(\mathcal{P}, u)$ in the following way. First, we project all points of \mathcal{P} on $\rho(u)$ and obtain the coordinate $\langle u, v \rangle$ for each $v \in \mathcal{P}$. Second, we sort all points of \mathcal{P} by the coordinates of their projections on $\rho(u)$. Then, the values $\text{Pr}^R(v, u)$ for all points $v \in \mathcal{P}$ can be obtained in $O(n)$ time by considering the projection points on $\rho(u)$ from right to left. Finally, $f(\mathcal{P}, u)$ can be computed in additional $O(n)$ time. Therefore, the total time for computing $f(\mathcal{P}, u)$ is $O(n \log n)$, which is dominated by the sorting. If the points of \mathcal{P} are given sorted along the direction u , then we can avoid the sorting step and compute $f(\mathcal{P}, u)$ in overall $O(n)$ time.

It remains to compute $\nabla f(\mathcal{P}, u)$. Recall that $\nabla f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u) v$ by the proof of Lemma 12. Note that the above has already computed $\text{Pr}^R(v, u)$ for all points $v \in \mathcal{P}$. Therefore, $\nabla f(\mathcal{P}, u)$ can be computed in additional $O(n)$ time. The lemma thus follows. \square

Lemma 44. *We can build a data structure of $O(n^2)$ size in $O(n^2 \log n)$ time that can compute $\omega(\mathcal{P}, u)$, $f(\mathcal{P}, u)$, and $\nabla f(\mathcal{P}, u)$ in $O(\log n)$ time for any query direction u . Further, we can construct M explicitly in $O(n^2 \log n)$ time.*

Proof. Consider any direction u with $\|u\| = 1$. We follow the definitions and notations in the proof of Lemma 43. We first show how to build a data structure to compute $f(\mathcal{P}, u)$.

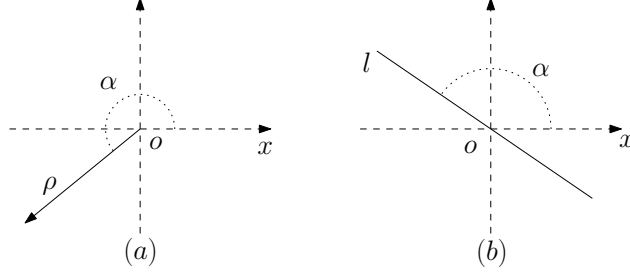


Figure 4: Illustrating the definition of the angle α of: (a) a ray ρ and (b) a line l .

Computing $f(\mathcal{P}, -u)$ can be done similarly. Again, after having $f(\mathcal{P}, u)$ and $f(\mathcal{P}, -u)$, $\omega(\mathcal{P}, u)$ can be computed immediately by $\omega(\mathcal{P}, u) = f(\mathcal{P}, u) - f(\mathcal{P}, -u)$.

Denote by o the origin. For any ray ρ through o in the plane, we refer to the *angle* of ρ as the angle α in $[0, 2\pi)$ such that after we rotate the x -axis around o counterclockwise by α the x -axis has the same direction as ρ (see Fig. 4(a)). For any (undirected) line l through o , we refer to the *angle* of l as the angle α in $[0, \pi)$ such that after we rotate the x -axis around o counterclockwise by α the x -axis is collinear with l .

Recall $\rho(u)$ is the ray through o with direction u . We define the *angle* of u as the angle of the ray $\rho(u)$, denoted by θ_u . For ease of discussion, we assume θ_u is in $[0, \pi)$ since the case $\theta_u \in [\pi, 2\pi)$ can be handled similarly.

We call the order of the points of \mathcal{P} sorted by the coordinates of their projections on the ray $\rho(u)$ the *canonical order* of \mathcal{P} with respect to u . An easy observation is that when we increase the angle u , the canonical order of \mathcal{P} does not change until u is perpendicular to a line containing two points of \mathcal{P} . There are $O(n^2)$ lines in the plane each of which contains two points of \mathcal{P} and the directions of these lines partition $[0, \pi)$ into $O(n^2)$ intervals such that if θ_u changes in each interval the canonical order of \mathcal{P} does not change. In the following, we show that for each of the above intervals, the value of $f(\mathcal{P}, u)$ is a function of the angle θ_u , and more specifically $f(\mathcal{P}, u) = a \cdot \cos(\theta_u) + b \cdot \sin(\theta_u)$ where a and b are constants when θ_u changes in the interval. As preprocessing for the lemma, we will compute the function $f(\mathcal{P}, u)$ for each interval; for each query direction u , we first find the interval that contains θ_u by binary search in $O(\log n)$ time and then obtain the value $f(\mathcal{P}, u)$ in constant time using the function for the interval. The details are given below.

For simplicity of discussion, we make a general position assumption that no three points of \mathcal{P} are collinear. For any two points v and v' in \mathcal{P} , let $\beta(v, v')$ denote the angle of the line perpendicular to the line containing v and v' , and we also say $\beta(v, v')$ is *defined* by v and v' . We sort all $O(n^2)$ angles $\beta(v, v')$ for $v, v' \in \mathcal{P}$ in increasing order, and let $\beta_1, \beta_2, \dots, \beta_h$ be the sorted list with $h = O(n^2)$. For simplicity, let $\beta_0 = 0$ and $\beta_{h+1} = \pi$. These angles partition $[0, \pi)$ into $h + 1$ intervals. Consider an interval $I_i = (\beta_i, \beta_{i+1})$ for any $0 \leq i \leq h$. Below we compute the function $f(\mathcal{P}, u) = a \cdot \cos(\theta_u) + b \cdot \sin(\theta_u)$ for $\theta_u \in (\beta_i, \beta_{i+1})$. Again, note that when θ_u changes in I_i , the canonical order of \mathcal{P} does not change.

According to the proof of Lemma 43, $f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \Pr^R(v, u) \langle v, u \rangle$. Since the canonical order of \mathcal{P} does not change for any $\theta_u \in I_i$, for any $v \in \mathcal{P}$, $\Pr^R(v, u)$ is a constant when θ_u

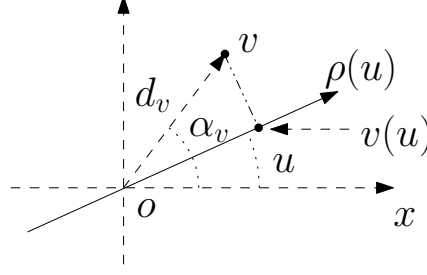


Figure 5: Illustrating the computation of the coordinate $x(v, u)$ on $l(u)$: $v(u)$ is the perpendicular projection of v on $l(u)$. The length of \overline{ov} is d_v .

changes in I_i . Next, we consider the coordinate $\langle v, u \rangle$ on $\rho(u)$.

For each point $v \in \mathcal{P}$, let α_v be the angle of the ray originating from o and containing v (i.e., directed from o to v), and let d_v be the length of the line segment \overline{ov} . Note that α_v and d_v are fixed for the input. Then, we have (see Fig. 5)

$$\langle v, u \rangle = d_v \cdot \cos(\alpha_v - \theta_u) = d_v \cdot \cos(\alpha_v) \cdot \cos(\theta_u) + d_v \cdot \sin(\alpha_v) \cdot \sin(\theta_u).$$

Hence, we have the following

$$\begin{aligned} f(\mathcal{P}, u) &= \sum_{v \in \mathcal{P}} \Pr^R(v, u) \langle v, u \rangle \\ &= \sum_{v \in \mathcal{P}} \Pr^R(v, u) \cdot d_v \cdot [\cos(\alpha_v) \cdot \cos(\theta_u) + \sin(\alpha_v) \cdot \sin(\theta_u)] \\ &= \cos(\theta_u) \cdot \left[\sum_{v \in \mathcal{P}} \Pr^R(v, u) \cdot d_v \cdot \cos(\alpha_v) \right] + \sin(\theta_u) \cdot \left[\sum_{v \in \mathcal{P}} \Pr^R(v, u) \cdot d_v \cdot \sin(\alpha_v) \right]. \end{aligned}$$

Let $a = \sum_{v \in \mathcal{P}} \Pr^R(v, u) \cdot d_v \cdot \cos(\alpha_v)$ and $b = \sum_{v \in \mathcal{P}} \Pr^R(v, u) \cdot d_v \cdot \sin(\alpha_v)$. Hence, a and b are constants when θ_u changes in I_i . Then, we have $f(\mathcal{P}, u) = a \cdot \cos(\theta_u) + b \cdot \sin(\theta_u)$, for any $\theta_u \in I_i$. Therefore, if we know the two values a and b , we can compute $f(\mathcal{P}, u)$ in constant time for any direction $\theta_u \in I_i$.

In the sequel, we show that we can compute a and b for all intervals $I_i = (\beta_i, \beta_{i+1})$ with $i = 0, 1, \dots, h$ in $O(n^2)$ time. For each interval I_i , we use $a(I_i)$ and $b(I_i)$ to denote the corresponding a and b respectively for the interval I_i .

Suppose we have computed $a(I_i)$ and $b(I_i)$ for the interval I_i , and also suppose we have computed the value $\Pr^R(v, u)$ for each point $v \in \mathcal{P}$ when $\theta_u \in I_i$ (note that $\Pr^R(v, u)$ is a constant for any $\theta_u \in I_i$). Initially, we can compute these values for the interval I_0 in $O(n \log n)$ time by Lemma 43. Below, we show that we can obtain $a(I_{i+1})$ and $b(I_{i+1})$ in constant time, based on the above values maintained for I_i .

Recall that $I_i = (\beta_i, \beta_{i+1})$ and $I_{i+1} = (\beta_{i+1}, \beta_{i+2})$. Suppose the angle β_{i+1} is defined by the two points v_1 and v_2 of \mathcal{P} . In other words, β_{i+1} is the angle of the line perpendicular to the line through v_1 and v_2 . If we increase the angle θ_u in (β_i, β_{i+2}) , the canonical order of \mathcal{P} does not change except that v_1 and v_2 exchange their order when θ_u passes the value

β_{i+1} . Therefore, for each point $v \in \mathcal{P} \setminus \{v_1, v_2\}$, the value $\text{Pr}^R(v, u)$ is a constant for any $\theta_u \in (\beta_i, \beta_{i+2})$. Based on this observation, we can compute $a(I_{i+1})$ in the following way.

We first analyze the change of the values $\text{Pr}^R(v_1, u)$ and $\text{Pr}^R(v_2, u)$ when θ_u changes from I_i to I_{i+1} . Let u and u' be any two directions such that θ_u is in I_i and $\theta_{u'}$ is in I_{i+1} . Without loss of generality, we assume $\langle v_1, u \rangle < \langle v_2, u \rangle$, and thus, $\langle v_1, u' \rangle > \langle v_2, u' \rangle$ since v_1 and v_2 exchange their order. Observe that $\text{Pr}^R(v_1, u') = \text{Pr}^R(v_1, u)/(1 - p_{v_2})$ and $\text{Pr}^R(v_2, u') = \text{Pr}^R(v_2, u) \cdot (1 - p_{v_1})$. Thus, we can obtain $\text{Pr}^R(v_1, u')$ and $\text{Pr}^R(v_2, u')$ in constant time since we already maintain $\text{Pr}^R(v_1, u)$ and $\text{Pr}^R(v_2, u)$. Consequently, we have

$$\begin{aligned} a(I_{i+1}) &= a(I_i) - [\text{Pr}^R(v_1, u)d_{v_1} \cos(\alpha_{v_1}) + \text{Pr}^R(v_1, u)d_{v_2} \cos(\alpha_{v_2})] \\ &\quad + [\text{Pr}^R(v_1, u')d_{v_1} \cos(\alpha_{v_1}) + \text{Pr}^R(v_1, u')d_{v_2} \cos(\alpha_{v_2})] \\ &= a(I_i) + d_{v_1} \cos(\alpha_{v_1}) \cdot [\text{Pr}^R(v_1, u') - \text{Pr}^R(v_1, u)] + d_{v_2} \cos(\alpha_{v_2}) \cdot [\text{Pr}^R(v_2, u') - \text{Pr}^R(v_2, u)]. \end{aligned} \quad (15)$$

Hence, after we compute $\text{Pr}^R(v_1, u')$ and $\text{Pr}^R(v_2, u')$, we can obtain $a(I_{i+1})$ in constant time.

Similarly, we can obtain $b(I_{i+1})$ in constant time. Also, the values $\text{Pr}^R(v_1, u)$ and $\text{Pr}^R(v_2, u)$ are updated for $\theta_u \in I_{i+1}$.

In summary, after the $O(n^2)$ angles $\beta(v, v')$ are sorted in $O(n^2 \log n)$ time, the above computes the functions $f(\mathcal{P}, u) = a(I_i) \cdot \cos(\theta_u) + b(I_i) \cdot \sin(\theta_u)$ for all intervals I_i with $i = 0, 1, \dots, h$, in additional $O(n^2)$ time. This finishes our preprocessing.

Consider any query direction u . By binary search, we first find the two angles β_i and β_{i+1} such that $\beta_i \leq \theta_u < \beta_{i+1}$. If $\beta_i \neq \theta_u$, then θ_u is in I_i and we can use the function $f(\mathcal{P}, u) = a(I_i) \cos(\theta_u) + b(I_i) \sin(\theta_u)$ to compute $f(\mathcal{P}, u)$ in constant time. If $\beta_i = \theta_u$, then the function $f(\mathcal{P}, u) = a(I_i) \cos(\theta_u) + b(I_i) \sin(\theta_u)$ still gives the correct value of $f(\mathcal{P}, u)$ since when $\theta_u = \beta_i$ the projections of the two points of \mathcal{P} defining β_i on $\rho(u)$ overlap and we can still consider the canonical order of \mathcal{P} for $\theta_u = \beta_i$ the same as that for $\theta_u \in I_i$. Hence, the query time is $O(\log n)$.

Next, we show how to compute $\nabla f(\mathcal{P}, u)$. Recall that $\nabla f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u)v$ by the proof of Lemma 12. As preprocessing, we compute the value $\sum_{v \in \mathcal{P}} \text{Pr}^R(v, u)v$ for each interval $\theta_u \in (\beta_i, \beta_{i+1})$ for $i = 0, 1, \dots, h$. This can be done in $O(n^2)$ time (after we sort all angles), by using the similar idea as above. Specifically, suppose we already have $\nabla f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u)v$ for $\theta_u \in (\beta_i, \beta_{i+1})$; then we can compute $\nabla f(\mathcal{P}, u) = \sum_{v \in \mathcal{P}} \text{Pr}^R(v, u)v$ for $\theta_u \in (\beta_{i+1}, \beta_{i+2})$ in constant time. This is because when θ_u changes from (β_i, β_{i+1}) to $(\beta_{i+1}, \beta_{i+2})$, $\text{Pr}^R(v, u)$ does not change for any $v \in \mathcal{P} \setminus \{v_1, v_2\}$ and $\text{Pr}^R(v, u)$ for $v \in \{v_1, v_2\}$ can be updated in constant time, as shown above. Due to the above preprocessing, given any direction u , we can compute $\nabla f(\mathcal{P}, u)$ in $O(\log n)$ time by binary search, similar to that of computing $f(\mathcal{P}, u)$. Further, according to Lemma 12, the above preprocessing essentially computes M , in totally $O(n^2 \log n)$ time. The lemma thus follows. \square

D.2 Computing Expected Width for Locational Uncertainty

In this setting let \mathcal{P} be a set of n uncertain points each taking one of several locations from a set of m locations in S . The probability that a point $v \in \mathcal{P}$ is in location $s \in S$ is denoted $p_{v,s}$.

To simplify analysis and discussion, we assume each location $s \in S$ only has the potential to be realized by any one uncertain point $v \in \mathcal{P}$.

We now replicate the lemmas in the previous section for this setting. We use the same notation and structure when possible.

Lemma 45. *For any direction u , we can compute $\omega(\mathcal{P}, u)$, $f(\mathcal{P}, u)$, and $\nabla f(\mathcal{P}, u)$ in $O(m \log m)$ time; if the points of S are already sorted along the direction u , then we can compute them in $O(m)$ time.*

Proof. Again, we first compute $f(\mathcal{P}, u)$ since $\omega(\mathcal{P}, u) = f(\mathcal{P}, u) - f(\mathcal{P}, -u)$ and $f(\mathcal{P}, -u)$ can be computed similarly.

We follow the structure and proof of Lemma 43 and just note the changes. The first change is that we need to keep a bit more structure since there is now dependence between the different locations of each uncertain point v . Denote by $S^R(s, u)$ the subset of $s' \in S$ such that $\langle s', u \rangle > \langle s, u \rangle$ and denote by $\text{Pr}_\emptyset^R(s, u)$ as the probability that no point $v \in \mathcal{P}$ appears at a larger location than $s \in S$ along direction u . To describe this probability we first define a vector A_v indexed by s as $A_v[s] = 1 - \sum_{s' \in S^R(s, u)} p_{v, s'}$ as the probability that uncertain point v does not appear in any of its possible locations which are after s along direction u . Now we can define

$$\text{Pr}_\emptyset^R(s, u) = \prod_{v \in \mathcal{P}} A_v[s].$$

Finally, we define $\text{Pr}^R(s, v, u)$ as the probability that the largest point along u is uncertain point $v \in \mathcal{P}$ at location $s \in S$. This updates equation (14) to be

$$\text{Pr}^R(v, s, u) = p_{v, s} \cdot \text{Pr}_\emptyset^R(s, u) / A_v[s].$$

Note the two key differences. First we need to sum the probabilities for each location of v since they are mutually exclusive. Second, value $A_v[s]$ needs to be factored out of $\text{Pr}_\emptyset^R(s, u)$ because it is already accounted for in $p_{v, s}$ locating v at s , again since they are mutually exclusive.

It follows that $f(\mathcal{P}, u) = \sum_{s \in S, v \in \mathcal{P}} \text{Pr}^R(v, s, u) \langle s, u \rangle$.

To compute $f(\mathcal{P}, u)$ we again start by projecting all points P onto $\rho(u)$ obtaining coordinates $\langle u, s \rangle$ and sorting, if needed. This takes $O(m \log m)$ time. Given these coordinates, sorted, it now takes a bit more work in the locational setting to show that $f(\mathcal{P}, u)$ can be computed in $O(m)$ additional time. We focus on computing all m values $\text{Pr}^R(v, s, u)$; from there is straightforward to compute $f(\mathcal{P}, u)$ in $O(m)$ time.

We sweep over the points $s \in P$ from largest $\langle s, u \rangle$ value to smallest, and we maintain each $A_v[s]$ and $\text{Pr}_\emptyset^R(s, u)$ along the way. Given these, it is not hard to calculate $\text{Pr}^R(v, s', u)$ in constant time with $p_{v, s'}$ for $s' \in S$ as the next smallest value $\langle s', u \rangle$. The important observation is that we only need to update $A_v[s]^{\text{new}} = A_v[s]^{\text{old}} - p_{v, s}$ if $p_{v, s} > 0$ (which by assumption holds for only one $v \in \mathcal{P}$). Then $\text{Pr}_\emptyset^R(s, u)$ is updated by multiplying by $A_v[s]^{\text{new}} / A_v[s]^{\text{old}}$. Both operations can be done in constant time as needed to complete the proof.

It remains to compute $\nabla f(\mathcal{P}, u)$. It is not hard to see that in the locational model

$$\nabla f(\mathcal{P}, u) = \sum_{s \in S, v \in \mathcal{P}} \Pr^R(v, s, u) v.$$

Note that the above has already computed the m values $\Pr^R(v, s, u)$ for all $s \in S$ and $v \in \mathcal{P}$. Therefore, $\nabla f(\mathcal{P}, u)$ can be computed in additional $O(m)$ time. \square

Lemma 46. *We can build a data structure of $O(m^2)$ size in $O(m^2 \log m)$ time that can compute $\omega(\mathcal{P}, u)$, $f(\mathcal{P}, u)$, and $\nabla f(\mathcal{P}, u)$ in $O(\log m)$ time for any query direction u . Further, we can construct M explicitly in $O(m^2 \log m)$ time.*

Proof. Again we first discuss the case for computing $f(\mathcal{P}, u)$. For ease of discussion, we assume the angle θ_u is in $[0, \pi)$. We again follow the structure of Lemma 44. The geometry is largely the same, except that there are $h = O(m^2)$ angles $\beta_1, \beta_2, \dots, \beta_h$ since each pair $s, s' \in S$ now defines an angle $\beta(s, s')$. But it remains to compute $f(\mathcal{P}, u) = a \cdot \cos(\theta_u) + b \cdot \sin(\theta_u)$ for some constants a and b for any θ_u in each (β_i, β_{i+1}) . The argument is virtually the same, replacing $\Pr^R(v, u)$ with $\Pr^R(v, s, u)$.

It remains to show that we can calculate the constants $a(I_{i+1})$ and $b(I_{i+1})$ for an interval $I_{i+1} = (\beta_{i+1}, \beta_{i+2})$ efficiently, given the values for interval $I_i = (\beta_i, \beta_{i+1})$. Assume β_{i+1} is defined for two points $s_1, s_2 \in S$, where $\langle s_1, u \rangle < \langle s_2, u \rangle$ for $\theta_u \in I_i$ and $\langle s_1, u' \rangle > \langle s_2, u' \rangle$ for $\theta_{u'} \in I_{i+1}$. By definition, the ordering among all other pairs of points is unchanged within (β_i, β_{i+2}) . Let only v_1 take location s_1 with positive probability and only v_2 take location s_2 with positive probability. We focus on the more general case where $v_1 \neq v_2$; when $v_1 = v_2$, it is easier to update.

We again focus on updating a and the algorithm for b is symmetric. By the $v_1 \neq v_2$ assumption $A_{v_1}[s_1]$ and $A_{v_2}[s_2]$ are unchanged in interval (β_i, β_{i+2}) . However from directions u to u' $A_{v_2}[s_1]$ increases by p_{v_2, s_2} and $A_{v_1}[s_2]$ decreases by p_{v_1, s_1} ; all other such values are unchanged. Let $A_v[s]^u$ denote the value in direction u . Hence

$$\Pr_{\emptyset}^R(s_1, u') = \Pr_{\emptyset}^R(s_1, u) \frac{A_{v_2}^{u'}[s_1]}{A_{v_2}^u[s_1]},$$

and so if we can update $A_{v_2}[s_1]$, we can update $\Pr_{\emptyset}^R(s_1, u')$ in constant time. Only $A_{v_1}[s_2]$ and $\Pr_{\emptyset}^R(s_2, u')$ also need to be updated. And then

$$\Pr^R(v_1, s_1, u') = p_{v_1, s_1} \cdot \Pr_{\emptyset}^R(s_1, u') / A_{v_1}[s_1] = \Pr^R(v_1, s_1, u) \frac{\Pr_{\emptyset}^R(s_1, u')}{\Pr_{\emptyset}^R(s_1, u)}$$

can also be updated in constant time, and similar for $\Pr^R(v_2, s_2, u')$. Thus the only remaining difficulty is accessing and updating $A_{v_1}[s_2]$ and $A_{v_2}[s_1]$. We can easily do this for if we store the full size m array $A_v[\cdot]$ for each $v \in \mathcal{V}$. Note that this takes $O(m \cdot n)$ space, but since the output is a structure of size $O(m^2)$ and $n \leq m$, this is not prohibitive. (We note that these full arrays are not explicitly required in Lemma 45, which only requires $O(m)$ space.)

Finally, we can update $a(I_{i+1})$ from $a(I_i)$ similarly to equation (15) using $\text{Pr}^R(v, s, u)$ in place of $\text{Pr}^R(v, u)$. Thus in $O(m^2)$ time, after sorting all interval breakpoints in $O(m^2 \log m)$ time, we can build a data structure that allows calculation of $f(\mathcal{P}, u)$ for any u in $O(\log m)$ time.

Next, to compute $\nabla f(\mathcal{P}, u)$, recall that in the locational model $\nabla f(\mathcal{P}, u) = \sum_{s \in S, v \in \mathcal{P}} \text{Pr}^R(v, s, u)v$ by the proof of Lemma 12. As preprocessing, we compute the value $\sum_{s \in S, v \in \mathcal{P}} \text{Pr}^R(v, u)v$ for each interval $\theta_u \in (\beta_i, \beta_{i+1})$ for $i = 0, 1, \dots, h$. This can be done in $O(m^2)$ time (after we sort all angles), by using the similar idea as above. The argument is similar to that in Lemma 44 and we ignore the details. Due to the above preprocessing, given any direction u , we can compute $\nabla f(\mathcal{P}, u)$ in $O(\log m)$ time by binary search. Further, the above preprocessing essentially computes M , in totally $O(m^2 \log m)$ time. The lemma thus follows. \square